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LOCALIZABILITY AND CAUSAL PROPAGATION IN RELATIVISTIC QUANTUM MECHANICS

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We show that, in a relativistic quantum theory in which the mass shell is not sharp, and positive and negative energy states are admissable, causal propagation is possible, and Hegerfeldt's theorem can be avoided. The conditions under which this is true have simple physical interpretation.

Key words: relativistic quantum mechanics, causal propagation, localization.

Hegerfeldt [1] has shown that a wave function which represents a particle localized according to some reasonable definition (for example, in the sense of Newton and Wigner [2]), propagates noncausally under relativistic free evolution. This effect is due to the non-analyticity of the form of the energy as a function of momentum $E = \sqrt{\mathbf{p}^2 + m^2}$. The relativistic description of quantum states used implying this result assumes that the system has only positive energy and a definite sharp mass. In the following we shall use a manifestly covariant representation of quantum states in which these constraints are relaxed to a certain extent and show that causal propagation can be achieved.

Although non-causal propagation has been shown to occur for functions with rapid decrease [3] as well, we review here the argument of Hegerfeldt [1], for the case in which the initial state represented by $\phi(\mathbf{x},t)|_{t=0}$ has compact support in a region V, i.e.,

$$\langle \phi(\mathbf{x},0), N(V)\phi(\mathbf{x},0)\rangle = 1,$$

where $N(V)$ is the projection operator into states with support in V.

Causality implies that if, at time t_0 the particle state is localized in this way, then there is an r_t such that, at time $t > 0$, the wave function, when translated by $\mathbf{a}$, $|\mathbf{a}| > r_t$, has no support in V. It then follows that

$$\langle U(-\mathbf{a})U(t)\phi_0, N(V)U(-\mathbf{a})U(t)\phi_0\rangle = 0.$$

Since $N(V)^{\frac{1}{2}}$ is self-adjoint, one sees that the scalar product of the displaced wave function has to be orthogonal to the initial one, i.e.,

$$\int \frac{d^3p}{\sqrt{\mathbf{p}^2+m^2}}|\phi(\mathbf{p})|^2 \exp i(\mathbf{p}\cdot\mathbf{a} - \sqrt{\mathbf{p}^2+m^2}t) = 0, \qquad (1)$$

where $\phi(\mathbf{p})$ is the momentum representation of the initial state. The coefficient of $\exp i\mathbf{p}\cdot\mathbf{a}$ in the integral must therefore be an entire function of $\mathbf{p}$. This is clearly not possible for all t (see, however, the discussion of [4]).

In the framework that we shall use, the quantum state of a particle is described in terms of a wave function for events in spacetime $\psi_\tau(x)(x \equiv \mathbf{x},t)$, as proposed by Stueckelberg [5], Schwinger [6] and Feynman [7]. This function corresponds to the amplitude for finding a local event at the spacetime point x; it is parametrized by a universal "proper time" τ. Its evolution is governed by the equation [5]

$$i\frac{\partial\psi_\tau(x)}{\partial\tau} = K\psi_\tau(x), \qquad (2)$$

where K is an operator analogous to the Hamiltonian of the non-relativistic Schrödinger theory. The quantum mechanical equations corresponding to the classical Hamilton equations

$$\frac{dx^\mu}{d\tau} = \frac{\partial K}{\partial p_\mu}, \qquad \frac{dp^\mu}{d\tau} = -\frac{\partial K}{\partial x_\mu} \qquad (3)$$

are

$$\frac{dx^\mu}{d\tau} = i[K, x^\mu], \qquad \frac{dp^\mu}{d\tau} = i[K, p^\mu], \tag{4}$$

where the canonical commutation relations are

$$[q^\mu, p^\nu] = ig^{\mu\nu}. \tag{5}$$

The variables $(\mathbf{x}, t)$ are considered as dynamical variables of the theory; the complementary variables $(\mathbf{p}, E)$ are also independent dynamical variables and hence the particle is not restricted to a precise mass-shell. It is possible, nevertheless, to impose the condition that the distribution of $m^2 \equiv E^2 - \mathbf{p}^2$ is as sharp as one wishes.

The Fourier transform function $\psi_\tau(p)(p \equiv p^\mu)$ may have support on both positive and negative energy. The negative energy components can be understood according to (3) (for the Ehrenfest motion of the wave packet), where, for example, for the free particle with invariant Hamiltonian,

$$K = K_0 = \frac{p^\mu p_\mu}{2M}, \tag{6}$$

one obtains

$$\frac{dt}{d\tau} = \frac{E}{M}. \tag{7}$$

For $E < 0$, the motion of the wave packet is in the negative direction of t when τ increases. This was interpreted by Feynman and Stueckelberg as a representation of the antiparticle (one can show that the CPT conjugate of the wave function describes a particle with opposite p^μ and charge). The particle and antiparticle occur as different aspects of the same entity; wave functions with support on positive and negative energy may occur in superposition. Many subsequent investigations of the manifestly covariant quantum theory associated with this framework have been carried out [8,9,10,11].

As a function of τ, a free wave packet has the form

$$\psi_\tau(x) = \int e^{ip_\mu x^\mu} e^{-i\frac{p^\mu p_\mu}{2M}\tau} \psi(p) d^4p. \tag{8}$$

To discuss the question of causal propagation, we must define the locality properties of $\psi_\tau(x)$. This function corresponds to a wave packet in spacetime which moves with Ehrenfest motion along a diffusely defined world line with τ. What is in question is the the possibility of obtaining a positive response to a position measurement in some region at a specific time, independently of τ. This question can be formulated more precisely by recognizing that the occurrence of an event at $(\mathbf{x}, t)$ at some τ with probability density (on d^4x) $|\psi_\tau(\mathbf{x}, t)|^2$, implies the possibility of registering a signal at some space point at a given time, say t_0, which lies on a (virtual) extrapolating world line passing through the event $\mathbf{x}, t$. The operator whose spectrum provides this information is [8]

$$\mathbf{x}_{NW}(t_0) = \mathbf{x} - \frac{1}{2}\{t - t_0, \frac{\mathbf{p}}{E}\}, \tag{9}$$

where we have appended the subscript NW, to indicate that this operator corresponds to the Newton-Wigner position operator [2] for each value of $m = \sqrt{E^2 - \mathbf{p}^2}$, in a mass shell decomposition. In fact,

$$\begin{aligned}\int d^4p\psi^*(p)(i\frac{\partial}{\partial\mathbf{p}} - \frac{1}{2}\{\frac{\mathbf{p}}{p_0}, t - t_0\})\psi(p) = \\ \int dm^2 \int \frac{d^3\mathbf{p}}{p_0}\psi^*(p)[i\frac{\partial}{\partial\mathbf{p}} - i\frac{\mathbf{p}}{2p_0{}^2} + t_0\frac{\mathbf{p}}{p_0}]\psi(p),\end{aligned} \tag{10}$$

where we have changed variables and redefined the derivative $\frac{\partial}{\partial\mathbf{p}}$, so that it acts on $p^0 = \pm\sqrt{\mathbf{p}^2 + m^2}$. The expression in the square brackets is the momentum representation of the Newton-Wigner operator at t_0.

We now define the projection operator $N_{t_0}(V)$ for which

$$(\psi_\tau, N_{t_0}(V)\psi_\tau)$$

is the probability for the Newton-Wigner position to lie in the space volume V, at time t_0. This probability is independent of τ, when ψ_τ develops according to the free evolution (6), since

$$[\mathbf{x}_{NW}(t_0), K_0] = 0. \tag{11}$$

The projection operator $N_{t_0}(V)$ provides the characterization of a position measurement independent of τ. We then follow Hegerfeldt [1] to define a criterion of causal propagation for the case of a state with compact support (in the Newton-Wigner sense that we have defined above) at some initial time.

The procedure previously followed (to obtain (1)), now appears in the following form: Let (independently of τ)

$$\psi_{\Delta\mathbf{x}\Delta t}(p) = e^{i\mathbf{p}\cdot\Delta\mathbf{x}} e^{-iE\Delta t}\psi(p). \tag{12}$$

where $\psi_{\Delta\mathbf{x}\Delta t}$ is the space time translation of $\psi(p)$, and let $\psi(p)$ have Newton-Wigner support in $|\mathbf{x}_{NW}(t_0)| < R_{t_0}$, i.e.,

$$N_{t_0}(V)\psi(p) = \psi(p). \tag{13}$$

Let us now study the position resulting from a measurement at a time $t_1 > t_0$. As for (1), it follows that if this wave function has the property of causal propagation in space-time, then there should exist at a later time t_1, a $\Delta x(t_1) > R_t$, such that

$$N_{t_1}(V)e^{i\mathbf{p}\cdot\Delta\mathbf{x}}\psi(p) = 0$$

where $\Delta t = t_1 - t_0$ and therefore

$$\int d^4p\psi(p)^* e^{i\mathbf{p}\cdot\Delta\mathbf{x}-iE\Delta t}\psi(p) = 0. \tag{14}$$

for $|\Delta\mathbf{x}| > R_t$. The Fourier transform then implies that

$$F(\mathbf{p}) = \int dE\psi(p)^* e^{-iE\Delta t}\psi(p). \tag{15}$$

must be entire in $\mathbf{p}$.

If we assume that the support of the wave function admits an integration over E that is independent of $\mathbf{p}$, e.g., if the wave function has a factorized form, then there would be no explicit non-analytic behavior in $\mathbf{p}$, and hence no contradiction with causality. If we define the variable $m^2 = E^2 - \mathbf{p}^2$, it is clear that $\psi(p)$ would, in this case, contain tachyonic components. In the covariant quantum theory we are using, the wave function could develop tachyonic components (corresponding to pair creation and annihilation) during the process of localization.

On the other hand, it is possible to construct an analytic $F(\mathbf{p})$ with strict causal propagation, and no tachyonic components. To do this, we change the integration variable from E to m^2; setting $E^2 = \mathbf{p}^2 + m^2$, one obtains

$$F(\mathbf{p}) = \int \frac{dm^2}{2\sqrt{\mathbf{p}+m^2}}[|\psi^-(p)|^2 e^{i\sqrt{\mathbf{p}^2+m^2}\Delta t} + |\psi^+(p)|^2 e^{-i\sqrt{\mathbf{p}^2+m^2}\Delta t}], \tag{16}$$

where $\psi^\pm$ contain positive and negative energies respectively. We shall use to represent these functions, the generalized eigenfunctions of the Newton-Wigner operator[2], which are the solutions of

$$i(\partial_\mathbf{p} + \frac{\mathbf{p}}{E}\partial_E - \frac{\mathbf{p}}{2E^2})\psi_{NW}(\mathbf{p}, E) = \mathbf{x_0}\psi_{NW}(\mathbf{p}, E), \tag{17}$$

i.e.,

$$\psi_{NW}(\mathbf{p}, E) = \pm\sqrt{|E|}e^{i\mathbf{p}\cdot\mathbf{x_0}}\chi_\pm(E^2 - \mathbf{p}^2), \tag{18}$$

for $t_0 = 0$ (for $t_0 \neq 0$, one must multiply by e^{-iEt_0}). If we assume that the initially localized wave function is a superposition of such localized functions, within the domain described by $N_{t_0}(V)$, they are in fact solutions of $N_{t_0}(V)\psi = \psi$.

The balance of positive and negative energy components in the localized wave function, with which we test the notion of causal propagation, can be interpreted in terms of the action of a localizing filter. Passage through a position filter, corresponding to localization of a particle, can be thought of as the successive annihilation and creation of the particle by the filter. The annihilation of the particle is, however, equivalent to the creation of a particle in a negative energy state, and conversely; hence the process of testing for localization should result in an equally weighted occurrence of positive and negative energy parts of the wave function (the notion of filtering will be discussed further elsewhere). In this case $|\chi_+|^2 = |\chi_-|^2$, and one obtains,

$$F(\mathbf{p}) = \frac{1}{2}\int dm^2 cos(\sqrt{\mathbf{p}^2 + m^2}t)\chi^2(m^2). \tag{19}$$

The function $\chi(m^2)$ can have a very concentrated support; we see that $F(\mathbf{p})$ is indeed analytic since unlike the exponent of the square root [1], $cos(\sqrt{\mathbf{p}^2 + m^2}t)$ is analytic for every t.

To the extent to which causal propagation is observed, we see that the wave function must carry tachyonic components (if non-analytic behavior, due to branch points caused by the restriction to $E^2 - \mathbf{p}^2 \geq 0$ is not removed by the integration over E), or a balance of positive and negative energies. Eberhardt and Ross [12] have argued that acausal effects must be suppressed in QFT, due to the vanishing of the commutator of local observables, in space-like regions. The balance of $\pm$ energies in the localized wave-function is analogous to the two terms of the commutator.

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NOTE

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