Explicit illustration of causality violation: Noncausal relativistic wave-packet evolution

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A class of functions whose relativistic evolution is analytically solvable is found. These wave functions are initially localized (by a Newton-Wigner definition) and their spreading through time is investigated. Using these wave packets an explicit example of the violation of relativistic causality (in the spirit of Hegerfeldt's theorem) is presented. Some detailed qualitative features of relativistic quantum behavior are displayed.

I. INTRODUCTION

Hegerfeldt proved,[1,2] on very general grounds, a surprising theorem concerning the unavoidable noncausal behavior of a relativistic quantum particles. This theorem is applicable to any reasonable definition of probability density. We present in this paper an explicit example of this phenomena using a Newton-Wigner definition of probability.

This example is found through investigation of spreading of relativistic wave packets. The time evolution of one class of wave packets can be obtained analytically.

We consider a relativistic quantum theory in which the one-particle sector is independent of other sectors. More specifically, let us consider (for the sake of simplicity) the free relativistic theory of spinless particles described by the Hamiltonian

$$\hat{H}=\int \omega(k)\hat{a}_k^\dagger \hat{a}_k \ , \tag{1}$$

where

$$\omega(k)=(k^2+m^2)^{1/2} \ . \tag{2}$$

The one-particle sector is spanned by the basis elements $|k\rangle$ normalized as

$$\langle k \mid k'\rangle=\delta(k-k') \ . \tag{3}$$

An arbitrary one-particle state vector is

$$|\psi\rangle=\int \psi(k)|k\rangle \ . \tag{4}$$

The normalized function $\psi(k)$ is the analog of the Schrödinger wave function in nonrelativistic quantum mechanics and the probability density in momentum space,

$$\rho(k)=|\psi(k)|^2 \ , \tag{5}$$

gives a probabilistic interpretation in momentum space. According to Newton and Wigner,[3] the function

$$\psi(x)=(2\pi)^{-1/2}\int e^{-ikx}\psi(k) \tag{6}$$

is the probability amplitude in configuration space and the probability to find a particle at the place x is

$$\rho(x)=|\psi(x)|^2 \ . \tag{7}$$

The function $\psi(x)$ is not covariant but it is related to the covariant wave function by the Foldy-Wouthuysen transformation.[4,5] The time evolution of $\psi(x)$ derived from $\hat{H}$ is given by

$$i\frac{\partial}{\partial t}\psi_t(x)=(-\Delta+m^2)^{1/2}\psi_t(x) \tag{8}$$

that replaces the Schrödinger equation in configuration space.

This linear equation is not an ordinary differential equation, but in Fourier space it diagonalizes

$$i\frac{\partial}{\partial t}\psi_t(k)=\omega(k)\psi_t(k) \ . \tag{9}$$

We shall find in Sec. II an exact solution of Eq. (8).

This solution is used in Sec. III to give an explicit example of the violation of causality which was expected in the view of Hegerfeldt's theorem. In fact we find a causality violation even for wave functions which are not exponentially decaying (in the massless case) and which are not covered by the assumptions of Hegerfeldt's theorem.[2]

We also observe that when $t\to\infty$ the effect of the violation of causality disappears in accord with Ruijsennaar's theorem.[6]

II. RELATIVISTIC WAVE-PACKET EVOLUTION

For any equation of the type

$$i\frac{\partial}{\partial t}\psi_t(x)=\omega\left[-i\frac{\partial}{\partial x}\right]\psi_t(x) \tag{10}$$

there exists a set of states whose evolution can be easily found. Those are wave functions of the special form

$$\psi^\gamma(k)=\text{const}\times e^{-\gamma\omega(k)} \tag{11}$$

for which, according to Eq. (9),

$$\psi_t^\gamma(k)=\text{const}\times e^{-\gamma(t)\omega(k)} \ , \tag{12}$$

where

$$\gamma(t)=\gamma+it \ . \tag{13}$$

In the nonrelativistic case

$$\omega_{nr}(k)=\frac{k^2}{2m} \tag{14}$$

and this procedure results in the well-known Gaussian wave packets:[7]

$$\psi_t^\gamma(k)=\left[\frac{\gamma}{\pi m}\right]^{1/4} e^{-\gamma(t)\omega_{nr}(k)} . \tag{15}$$

In configuration space the nonrelativistic wave equation is also a Gaussian:

$$\psi_t^\gamma(x)=\left[\frac{\gamma}{\pi m}\right]^{-1/4} \exp\left[-\frac{mx^2}{2\gamma(t)}\right] . \tag{16}$$

The Gaussian width increases according to

$$(\Delta x)^2=\langle\psi\,|\,x^2\,|\,\psi\rangle=\left[\frac{1}{m}\right][\gamma^*(t)\gamma(t)]^{1/2} . \tag{17}$$

In the relativistic case $\omega(k)$ is given by Eq. (2) and consequently the momentum-space wave function is

$$\psi_t^\gamma(k)=[2mK_1(2m\gamma)]^{-1/2}\exp[-\gamma(t)(k^2+m^2)^{1/2}] . \tag{18}$$

The configuration-space wave function is

$$\psi_t^\gamma(x)=\{2m[x^2+\gamma^2(t)]K_1(2m\gamma)\}^{-1/2} \times\gamma(t)K_1(2m[x^2+\gamma(t)]^{1/2}) , \tag{19}$$

where K_1 is the Hankel function and $\gamma(t)$ is still given by Eq. (13).

The case $m=0$ was already considered in Ref. 5. In this case the wave function is simply a Lorenzian:

$$\psi_t^\gamma(x)=\left[\frac{2\gamma}{\pi}\right]^{1/2}\frac{\gamma^{1/2}\gamma(t)}{x^2+\gamma(t)^2} . \tag{20}$$

In Figs. 1 and 2 the evolution of the probability density is shown for two different cases: In Fig. 1 the width of the initial wave packet is smaller than the Compton wavelength and in Fig. 2 the width of the initial wave packet is larger than the Compton wavelength (x, t, and γ are given in units of Compton wavelength).

We observe that when the width is larger than the Compton wavelength the behavior resembles the nonrelativistic case: the probability density has a peak at the center of the initial wave packet but the width increases with time. On the other hand, when, at $t=0$, the particle is localized within the Compton wavelength, we observe a qualitative change of the picture: the probability density at $t>0$ does not "remember" the original density, but it concentrates mainly around the light cone $\{(t,x);x=ct\}$. The probability density at the original position decreases with time. In spite of the finite mass $m>0$, this case resembles the ultrarelativistic case $m=0$, studied in Ref. 5. This is expected because in this case the mass is small compared with the characteristic momentum.

III. VIOLATION OF CAUSALITY IN RELATIVISTIC QUANTUM MECHANICS

We turn now to the investigation of the properties of the probability distribution $\rho_t(x)$ given by Eq. (19).

Hegerfeldt proved on very general grounds two surprising theorems concerning the evolution of relativistic localized wave packets. His first theorem[1] states that if at the time $t=0$ the wave function is localized in the compact region $x<a$ then at any later time $t>0$ the probability to find the particle outside the light cone $\{(t,x);x>a+ct\}$ does not vanish. This property conflicts with the intuitive notion of relativistic causality. Hegerfeldt further generalized the noncausality property to the evolution of wave packets that are not precisely localized inside a compact region of space, but have exponential tails.[2] More specifically (in the case of Newton-Wigner probability interpretation) this theorem applies to wave functions satisfying a condition

$$\int dx\,\rho_{t=0}(x)<K\exp(-K'a) , \tag{21}$$

where K is a positive constant and $K'>2m$.

Hegerfeldt's second theorem ensures that, at every

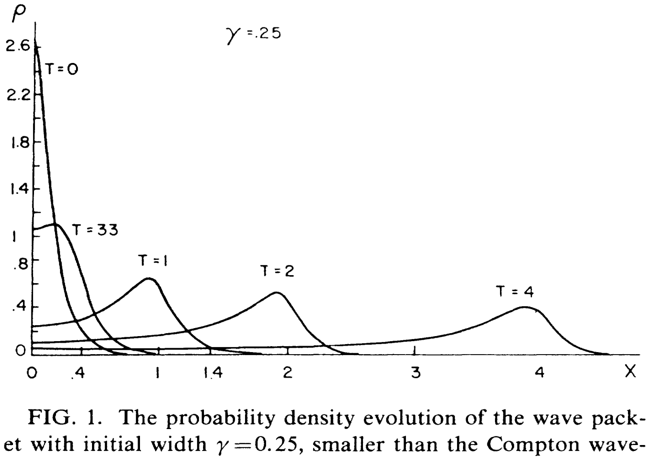

FIG. 1. The probability density evolution of the wave packet with initial width $\gamma=0.25$, smaller than the Compton wavelength (x, t, and γ are given in units of Compton wavelength).

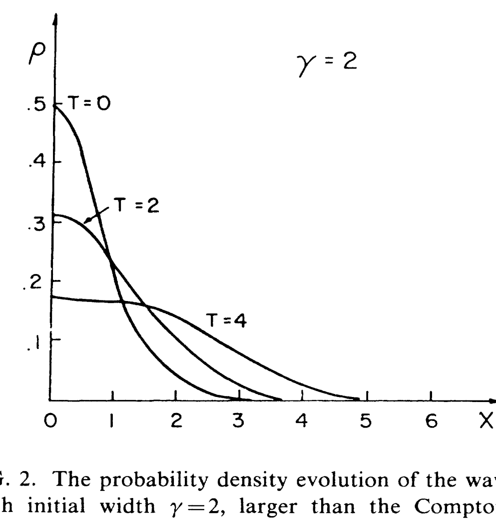

FIG. 2. The probability density evolution of the wave packet with initial width $\gamma=2$, larger than the Compton wavelength (x, t, and γ are given in units of Compton wavelength).

time $t>0$, a region $a<x<b$ exists, such that the probability to find the particle in it

$$\rho_t[a,b]=\int_{a<x<b} dx\, \rho_t(x) \tag{22}$$

is greater than the probability to find the particle in the region $a-ct<x<b+ct$ at time $t=0$:

$$\rho_{t=0}[a-ct,b+ct]<\rho_t[a,b] \ . \tag{23}$$

The region $\{(0,x);a-ct<x<b+ct\}$ belongs to the past light cone of the region $\{(t,x);a<x<b\}$ and therefore the inequality (23) violates causality. The causality violation for wave packets with exponentially decaying probability density can be constructively verified by finding an example of a wave-function evolution satisfying inequality (23).

As such an explicit example of noncausal behavior, for the massive case, we will take the region $x>5$ at $t=1$, for the initial wave function of Eq. (19), with $\gamma=0.25$ (x,t, and γ are given in units of Compton wavelength $1/m$). As a rule, far enough from the center of the wave packet $x \gg ct$, the inequality (23) is satisfied.

The wave function of Eq. (19) has the asymptotic behavior

$$\psi^\gamma(x)\sim\frac{1}{x^2}e^{-m|x|}\left[1+O\left[\frac{1}{x}\right]\right] \tag{24}$$

and the probability density satisfies

$$\lim_{a\to\infty}\int_{|x|>a} dx\, \rho^\gamma(x)\sim\frac{1}{a^4}e^{-2ma} \ . \tag{25}$$

Therefore our initial wave function satisfies Eq. (20), for $K'=2m$, being the limit case to the requirement used in the proof of Hegerfeldt's theorem ($K'>2m$) (Ref. 2). (Our wave function is localized exponentially but falls slower, and therefore we will show that causality is violated also for a case in which the wave packet is exponentially localized with a decaying constant equal to $2m$.)

Another explicit example of causality violation is provided by the wave packet (20), in the massless case, $m=0$. In this case the probability density is

$$\rho(x)=\frac{2}{\pi}\frac{\gamma(\gamma^2+t^2)}{|\gamma^2(t)+x^2|^2} \ . \tag{26}$$

The asymptotic behavior is not exponential but inverse fourth power, $\sim 1/x^4$, which is far less localized than assumed in the Hegerfeldt theorem.[2]

Nevertheless we observe a causality violation even in this case. Indeed, in this case,

$$\rho_t[a,\infty]=\frac{1}{2}-\frac{\gamma}{4\pi t}\ln\frac{(a+t)^2+\gamma^2}{(a-t)^2+\gamma^2} -\frac{1}{2\pi}\left[\frac{\arctan(a-t)}{\gamma}+\frac{\arctan(a+t)}{\gamma}\right] \tag{27}$$

and

$$\rho_{t=0}[a-t,\infty]=\frac{1}{2}-\frac{\gamma}{\pi}\frac{a-t}{(a-t)^2+\gamma^2}-\frac{1}{\pi}\frac{\arctan(a-t)}{\gamma} \ . \tag{28}$$

We checked that, for $t=\gamma$ and for $a=5\gamma$,

$$\rho_t(a,\infty)>\rho_{t=0}(a-t,\infty) \ . \tag{29}$$

Therefore we have again a causality violation even in a case which is power decaying.

We also investigated numerically the probability to find a massive particle outside the light cone of the origin $\{(t,x);x>ct\}$,

$$\rho_>^\gamma(t)=\int dx\, \rho_t^\gamma(x) \ , \tag{30}$$

and found the limits

$$\rho_>^\gamma(t)\underset{t\to\infty}{\longrightarrow}0 \ , \tag{31}$$

$$\rho_>^\gamma(t)\underset{\gamma\to 0}{\longrightarrow}\tfrac{1}{2} \ . \tag{32}$$

The first limit is an illustration of a general theorem proved by Ruijsenaars[6] which he calls the "asymptotical causality" theorem. Physically this theorem implies enormous difficulties in order to perform an experimental verification of noncausality. In this aspect our calculation permits an explicit estimate of the noncausal effect (not only the limit).

The second limit, Eq. (32), is a little bit surprising. It states that at any finite time, $t>0$, for a sufficiently narrow initial wave packet, the probability to find the particle inside a region $\{(t,x);x>ct\}$ of the light cone equals the probability to find the particle in a region outside the light cone $\{(t,x);x>ct\}$. This property can be proved analytically in the simple case $m=0$:

$$\rho_>^\gamma(t)=\frac{\gamma}{2\pi t}\ln\frac{4t^2+\gamma^2}{\gamma^2}+\frac{1}{\pi}\arctan\left[\frac{2t}{\gamma}\right]\underset{\gamma\to 0}{\longrightarrow}\frac{1}{2} \ . \tag{33}$$

In the place of the nonrelativistic formula for width increase of a wave packet, Eq. (17), we obtain, in the relativistic case,

$$(\Delta x)^2=\langle\psi|x^2|\psi\rangle=\gamma^*(t)\gamma(t)\left[1+\frac{m}{2}\frac{\int_{2\gamma}^{\infty}K_0(ms)ds}{K_1(2m\gamma)}\right] \ . \tag{34}$$

When $m=0$ we obtain

$$(\Delta x)^2=\gamma^*(t)\gamma(t) \ . \tag{35}$$

IV. DISCUSSION

We calculated the evolution of the wave function of Eq. (19) according to the free relativistic wave equation (8). The discussion was limited to the Newton-Wigner probabilistic interpretation of relativistic quantum mechanics. We observe that these dynamics conflicts with the following simple requirement of relativistic causality: the probability to find the particle inside a compact region R should not be larger than the probability to find the particle inside a region of the past light cone of R at an earlier time. This requirement should be a property of a causal theory.

Hegerfeldt proved that causality is violated in relativistic quantum mechanics, for any wave packet localized with exponentially decaying tails, for decay constant K bigger than twice the mass of the particle. Our example shows that this happens also for some distribution with decay constant K equal to twice the mass of the particle. Moreover, we found causality violation in the massless case for wave packets which are localized less than exponentially, namely, as an inverse power. This indicates that the assumption of exponentially decaying tails is not so crucial for causality violation.

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