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NOTE

Approximating Agreeing to Disagree Results with Common p-Beliefs*

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Monderer and Samet generalize Aumann's agreeing to disagree result for the case of beliefs. They show that if the posteriors of an event are "common p-belief" then they cannot differ by more than $2(1 - p)$. We provide a different proof of this result with a lower bound of $1 - p$. An example which attains this bound is provided.

Aumann's (1976) famous agreeing to disagree result states that the posteriors formed over an event X by rational players must coincide if they are commonly known. Monderer and Samet (1989) generalize this result for the case of beliefs. They define the notion "common p-belief" and show that if the posteriors of an event are common p-belief then they cannot differ by more than $2(1 - p)$. In this short note we provide a different proof of this result which enables us to obtain a bound of $1 - p$ over the difference between posteriors that can be sustained as common p-belief. We show that this is the best possible bound by presenting an example which attains it. As opposed to Monderer and Samet's result, our result imposes some restrictions on posteriors which are common p-belief for $p \leq \frac{1}{2}$.

Following Monderer and Samet's (1989) formulation, let I be a finite set of players and let (Ω, Σ, μ) be a probability space, where Ω is a space of

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states, Σ is a σ-algebra of events, and μ is a probability measure on Σ (to be interpreted as a common prior). For each $i \in I$, Π_i is a partition of Ω into measurable sets with positive measure. For $\omega \in \Omega$, denote by $\Pi_i(\omega)$ the element of Π_i containing ω. Π_i is interpreted as the information available to agent i; $\Pi_i(\omega)$ is the set of all states which are indistinguishable to i when ω occurs. We denote by $\mathscr{F}_i$ the σ-field generated by Π_i. That is, $\mathscr{F}_i$ consists of all unions of elements of Π_i. For $i \in I$, $E \in \Sigma$, $\omega \in \Omega$, and $p \in [0, 1]$, we say that "*i believes E with probability at least p at* ω," or simply "*i p-believes E at* ω" if $\mu(E \mid \Pi_i(\omega)) \geq p$. Denote by $B_i^p(E)$ the event "i p-believes E." That is,

$$B_i^p(E) = \{\omega : \mu(E \mid \Pi_i(\omega)) \geq p\}.$$

Notice that this is an event (i.e., measurable with respect to Σ). Moreover, for any $E \in \Sigma$, it is also measurable with respect to $\mathscr{F}_i$. It is straightforward to verify that B_i^p is monotone and satisfies $B_i^p B_i^p = B_i^p$. That is, for any $i \in I$, E, $F \in \Sigma$, and $p \in [0, 1]$, $E \subseteq F$ implies $B_i^p(E) \subseteq B_i^p(F)$, and $B_i^p(B_i^p(E)) = B_i^p(E)$.

DEFINITION. An event E is *evident p-belief* if for each $i \in I$

$$E \subseteq B_i^p(E).$$

DEFINITION. An event C *is common p-belief at* ω if there exists an evident p-belief event E such that $\omega \in E$ and for all $i \in I$

$$E \subseteq B_i^p(C).$$

We now turn to the agreeing to disagree result. Fix an event $X \in \Sigma$ and define functions f_i for all agents i by

$$f_i(\omega) = \mu(X \mid \Pi_i(\omega));$$

$f_i(\omega)$ is i's posterior probability of X. Let r_i, $i \in I$, be numbers in the interval $[0, 1]$, and consider the event

$$C = \bigcap_{i \in I} \{\omega \in \Omega : f_i(\omega) = r_i\}.$$

THEOREM. *If C is common p-belief at* $\omega \in \Omega$, *then* $|r_i - r_j| \leq 1 - p$ *for all* $i, j \in I$. *That is, if the posteriors of the event X are common p-belief at some* $\omega \in \Omega$, *then they cannot differ by more than* $1 - p$.

Proof. Suppose that C is common p-belief at $\omega \in \Omega$. There exists an evident p-belief event E such that $\omega \in E$ and $E \subseteq B_i^p(C)$ for all $i \in I$. Suppose also that $p > 0$ (the conclusion is trivial for $p = 0$). Define $\pi_i = B_i^p(E)$, and $\pi = \bigcap_{i \in I} \pi_i$. Since E is evident p-belief, $E \subseteq \bigcap_{i \in I} B_i^p(E)$ and $E \subseteq B_i^p(C)$. By applying monotonicity to the first expression, we obtain: (1) $\pi_i \subseteq B_i^p(\pi)$; and by applying monotonicity and $B_i^p B_i^p = B_i^p$ to the second expression, we obtain: (2) $\pi_i \subseteq B_i^p(C)$. Since B_i^p is measurable with respect to $\mathcal{F}_i$, π_i is a union of elements of Π_i and therefore by (1), $\mu(\pi \mid \pi_i) \geq p$. Hence for any $i, j \in I$: (3) $\mu(\pi_j \mid \pi_i) \geq p$. It is also the case that, $\mu(X \mid \pi_i) = r_i$. Otherwise, there exists an $\omega' \in \pi_i$ such that $\mu(X \mid \Pi_i(\omega')) \neq r_i$. For this ω', $\mu(C \mid \Pi_i(\omega')) = 0$, in contradiction to (2).

Now for any Y, $\mu(Y \mid \pi_i) \geq \mu(\pi_j \mid \pi_i)\mu(Y \mid \pi_i \cap \pi_j)$, and hence by (3) it follows that (4) $\mu(Y \mid \pi_i) \geq p\mu(Y \mid \pi_i \cap \pi_j)$. Substituting X for Y in (4) we conclude: (5) $r_i \geq p\mu(X \mid \pi_i \cap \pi_j)$. Substituting the complement of X for Y we obtain: (6) $r_i \leq p\mu(X \mid \pi_i \cap \pi_j) + (1 - p)$. By symmetry, (5) and (6) hold also for r_j and, therefore, $|r_i - r_j| \leq 1 - p$. Q.E.D.

EXAMPLE. Let $\Omega = \{1, 2, 3\}$. Pick any $0 \leq p \leq 1$ and set $\mu(\{1\}) = \mu(\{3\}) = (1 - p)/(2 - p)$ and $\mu(\{2\}) = p/(2 - p)$. Let $I = \{1, 2\}$ and suppose that $\Pi_1 = \{\{1, 2\}, \{3\}\}$ and $\Pi_2 = \{\{1\}, \{2, 3\}\}$. Consider the event $X = \{1, 2\}$. At $\omega = 2$, player 1's posterior is $\mu(X \mid \Pi_1(2)) = 1$ and player 2's posterior is $\mu(X \mid \Pi_2(2)) = p$. We claim that these posteriors are common p-belief at $\omega = 2$. Let $C = \{\omega \in \Omega\colon \mu(X \mid \Pi_1(2)) = 1 \text{ and } \mu(X \mid \Pi_2(2)) = p\}$. It is straightforward to verify that $E = \{2\}$ is evident p-belief and that $E \subseteq B_1^p(C) \cap B_2^p(C)$. It therefore follows that the bound obtained in the theorem cannot be improved in general.

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