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Common Beliefs and the Existence of Speculative Trade

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This paper shows that if rationality is not common knowledge, the no-trade theorem of Milgrom and Stokey fails to hold. We adopt Monderer and Samet's notion of common p-belief and show that when traders entertain doubts about the rationality of other traders, arbitrarily large volumes of trade as well as rationality may be common p-belief for a large p. Furthermore, rationality and trade may simultaneously be known to arbitrary large (but finite) degree. *Journal of Economic Theory* Classification numbers: C70, D82, D84.

1. INTRODUCTION

The information representation literature pioneered by Aumann (1976) has led to the surprising "no-trade theorem" of Milgrom and Stokey (1982) that states that given ex-ante Pareto efficient allocations, the arrival of new information will not induce further trade under the assumption that trade is common knowledge among traders. This counterintuitive result is generally interpreted as a "no-speculation" result. It is perplexing for two main reasons: first, it stands in stark disagreement with the general image of the stockbroker as, mostly, a speculant and, second, it is generally believed that without at least some amount of speculative trade we cannot explain the huge volumes of trade that we observe in security markets around the world. Ross (1989), for example, states "it is difficult to imagine that the volume of trade in securities markets has very much at all to do with the modest amount of trading required to accomplish the continuous and gradual portfolio rebalancing inherent in our current intertemporal models."

In general, no-trade results rely heavily on the strength of the common knowl-

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edge assumption. While common knowledge of various facts, most notably the model itself, is implicitly assumed in much of economic theory, this concept does not perfectly capture our ordinary, everyday notion of "knowledge." In particular, the absolute certainty which is implied by knowing, as opposed to believing, seems exaggerated. Especially when the subject of this knowledge may sometimes involve other people's thoughts or future actions. It therefore seems desirable to investigate whether an appealing weakening of the common knowledge assumption may generate speculative trade.

In this paper we attempt to formalize the intuition that people trade because they think they are smarter than others. We show that rationality may be "almost" common knowledge and still allow for trade to take place. Thus, there is no need to introduce "noise" or liquidity traders in order to justify speculative trade. Rational traders may speculate against each other because they believe that they are right while others may be wrong.

Several models of "bounded rationality" have already been proposed as explanations of the no-trade puzzle. Among the many contributions, a number of papers have focused on the axiomatic approach to common-knowledge (e.g., Bacharach, 1985; Geanakoplos, 1988; Samet, 1990). These papers focus on the axioms on the knowledge operator that characterize no-trade results. They show that to avoid no-speculation results partitional information structures must be replaced with information structures that represent less rational modes of reasoning. (For a simple example, see Rubinstein and Wolinsky, 1990.) A different approach is proposed by Dow, Madrigal, and Werlang (1990) that retains partitional information structures and generalizes Milgrom and Stokey's result as follows: given an ex-ante Pareto efficient allocation, there does not exist any other allocation which is ex-post Pareto dominating the first with respect to the traders' information and is commonly known. Thus, any motivation for trade is eliminated. Their result does not depend on the ex-ante and ex-post partitions, nor on the preferences of the traders which need not be concave (i.e., risk averse), or even increasing. More importantly, it allows traders to have different priors. Their result requires only that markets be complete and that the utility functions be state additive. In fact, they show that this requirement is also necessary. The intuition for their result is that when an event is commonly known (ex-post), this very fact is agreed upon by all traders, and thus can be incorporated into state-contingent trade ex-ante. However, they are able to provide an example where no-speculation fails, that is, where there is trade, under nonadditive probabilities.[1]

[1] This result seems to conflict with Morris (1994) that demonstrates the possibility of speculative trade when traders have different priors. The difference follows from the fact that ex-ante efficient allocations are defined differently in these two papers. Indeed, Morris notes that "if it is possible to make trade contingent on some event prior to the arrival of new information, the differences in prior beliefs about the event will not lead to trade" (Morris, 1994, p. 1339).

The literature thus offers two "bounded rationality" explanations for the no-trade puzzle: one suggests that information structures are nonpartitional, and the other—that beliefs are nonadditive. In both cases, the notion of "rationality" and the extent of deviation from it can only be described in a meta-model. That is, a statement such as "trader 1 beliefs that trader 2 is rational" has no formal content and, perhaps, no meaning at all.

By contrast, this paper suggests a model in which rationality of a trader is a well-defined event. We retain the classical assumptions regarding knowledge and beliefs; that is, the information structures are partitional and the priors are additive; furthermore, we retain the common prior assumption. We follow Aumann (1987) in that "rationality" is defined behaviorally as acting optimally given the available information and others actions. It is thus a well-defined even over which traders have beliefs (see Remark 5.7).

In the formal model, relaxing the assumption of common knowledge of rationality implies that there exist states of the world in which traders are indeed "irrational"; that is, traders behave suboptimally. However, we introduce these states of the world into the model in order to represent traders' *perception* of the world. We do not mean to imply that traders actually act suboptimally, only that they believe that others may do so. The states at which traders may irrational are thus not claimed to actually materialize. An illustration to the way we think about rationality is the following: Suppose that a trader arrives at his office every morning at 9 a.m. On the way, he goes over the morning news, and when he gets to his office, based on this information, he decides which trades to make. However, this trader may be late to work—in which case, important trading opportunities may be lost, or a junior trader might buy a certain asset instead of selling it. We think of the trader who arrives at his office on time as being "rational," and of the same trader, when he is late and a suboptimal action is taken, as being "irrational." The main point is that to explain trade, no trader actually needs to be late. It suffices that some traders suspect that others may suspect that others may suspect that a trader has not shown up on time.

Two alternative concepts have been proposed in the literature as possible weakenings of common knowledge. The first is Rubinstein's (1989) "almost common knowledge" that allows only a finite hierarchy of knowledge. However, Rubinstein shows in an example that is the game-theoretic formulation of the "coordination attack" problem (see Halpern, 1986) that even for arbitrarily high levels of knowledge, "almost common knowledge" does not approximate common knowledge in the sense that optimizing agents that are in a state of "almost common knowledge" about the game they play cannot behave as if the game is common knowledge; that is, they cannot play the "natural" Nash equilibria in this game. Monderer and Samet (1989) suggest yet another way of weakening the common knowledge assumption. Instead of truncating the knowledge hierarchy, they replace common knowledge with what they call common belief. This notion weakens "knowledge" to "belief" while retaining an infinite hierarchy of the latter. In this setup, they obtain an approximation to Aumann's (1976) "agreeing-to-disagree" result as well as continuity of the set of Nash equilibria "at common knowledge." They quantify the degree of belief by a parameter $p \in [0, 1]$ and show that if there is common p-belief of the "true" game being played, ϵ-optimizing players can almost mimic the behavior of players to whom the game played is common knowledge and therefore play one of its Nash equilibria.

In this paper we use Monderer and Samet's definition of common p-belief to develop a model of speculative trade in which there is a positive probability of common belief of arbitrarily large volumes of trade. Natural questions to ask then, are, is common p-belief the "right" concept to use? To what extent should we expect common p-belief of trade to arise in actual markets? And lastly, is common p-belief more plausible than common knowledge? For example, if the trade mechanism that we have in mind is trade that is finalized by a handshake, common knowledge is the natural concept to use. Each of the two traders shaking hands knows that trade takes place, knows that the other trader knows, and so on. The handshake is a means of instantaneously arriving at common knowledge.

Yet, we argue that common p-beliefs does arise naturally in many realistic settings. Consider, for example, a trade mechanism that operates as follows: two traders receive private information about a certain stock. Each decides whether to send a market maker a "buy" or a "sell" order. The market maker has no assets of his own, and he would therefore execute trade if and only if he can match "buy" and "sell" orders. Thus, when the traders choose their actions, they do not know that trade is about to take place, even if it eventually will and so the event of trade may be common p-belief without being common knowledge. Hence the notion of common p-belief is weaker than that of common knowledge, not only in a theoretical sense; many realistic trading mechanisms permit the former although not the latter. Of course, one may convincingly argue that in many realistic examples the infinite hierarchies of beliefs is also unreasonable. The main point of this paper is, however, that it *suffices* to weaken knowledge to belief, to account for trade.

The first step in establishing this result is the observation that, excluding risk management considerations, any speculation or bet originates from a state of conflicting opinions, or more colorfully, "it is difference of opinion that makes horse-races." (Mark Twain, 1992, p. 117) Namely, in the context of this paper, starting with a Pareto-efficient allocation, traders must have conflicting views regarding the outcome of a certain event in order to induce trade. Following the previous literature, we start with ex-ante Pareto-efficient markets in order to exclude risk management considerations, so that pure speculation is the only possible motivation for trade after the arrival of new information. Nevertheless, disagreement among traders, and even common belief of disagreement, intense as it might be, is not enough to create motivation for trade among completely rational traders. This observation enables us to strengthen Milgrom and Stokey's (1982) result; namely, starting with an ex-ante Pareto efficient allocation, a positive probability of a feasible and mutually acceptable trade implies that traders must be indifferent between this trade and the null trade (a similar result is stated in Geanakoplos, 1992). Consequently, the second step in constructing our argument is to allow traders to entertain some doubts concerning the perfect rationality of other traders. We then show that common belief of disagreement, together with the presence of doubts about the rationality of other traders, is sufficient for obtaining common belief of trade of arbitrarily large volumes among not-too-risk-averse traders. It is worth noting that the doubts about the rationality of other traders may be slight enough as to allow for common p-belief of rationality for p arbitrarily close to 1, together with "almost common knowledge" (as in Rubinstein, 1989) of arbitrarily high degree, while still being sufficient for our results.

In a related (but independently developed) paper, Sonsino (1995) also deals with common p-belief of trade. His main results are as following: first, suppose that a proposed trade B is fixed, then if the expected valuations of B are common p-belief then they cannot differ significantly (the bound on their difference, however, depends on B and can be arbitrarily large). Second, it is shown that as p approaches 1, there can be no common p-belief of trade. In his paper "trade" is implicitly assumed to occur whenever the traders have different conditional expectations for some trade B. Since Sonsino's model does not allow traders to be irrational at any state of the world, his results seem to be in contradiction to ours. The resolution to this apparent contradiction lies in the implicit notion of rationality; in Sonsino's paper, acts are not formally introduced into the model. Thus, when a trade B is offered, each trader simply computes its expectation, implicitly assuming it would be accepted by the other trader. By contrast, in the model presented above, each trader is explicitly aware of the possibility that the other trader may refuse to trade and, thus, the trade may be called off. That is, he is facing uncertainty both regarding the "objective" state of the world and other traders' actions. Thus, Sonsino (1995) implicitly assumes some type of irrationality in that his traders do not fully analyze the model as it is known to the outside observer. In this paper, irrationality is explicit and the model and be assumed to be commonly known. One way or another, our second theorem proves that common knowledge of "full" rationality precludes trade.

The rest of the paper is organized as follows. In Section 2, we present the fundamental results of the knowledge/belief literature, following Monderer and Samet (1989). In Section 3, we develop a model a model of speculative trade for the simpler case of risk neutral traders. In Section 4, we develop the general version of the model, allowing for risk-averse traders. Section 5 concludes.

2. COMMON BELIEF

Let I be a finite set of players and let (Ω, Σ, μ) be a probability space, where Ω is the space of states of the world, Σ is a σ-algebra of events, and μ is a nonatomic probability measure on Σ (to be interpreted as a common prior). For each $i \in I$, Π^i is a finite partition of Ω into measurable sets with positive measure. We use the notation $\Pi^i = \{\Pi^i_1, \Pi^i_2, \ldots, \Pi^i_{N_i}\}$. For $\omega \in \Omega$, denote by $\Pi^i(\omega)$ the element of Π^i containing ω. Π^i is interpreted as the information available to agent i; $\Pi^i(\omega)$ is the set of all states which are indistinguishable to i when ω occurs. We denote by $\mathcal{F}^i$ the (σ)-field generated by Π^i. That is, $\mathcal{F}^i$ consists of all unions of elements of Π^i. For $i \in I$, $E \in \Sigma$, $\omega \in \Omega$, and $p \in [0, 1]$, we say that "*i believes E with probability at least p at ω*," or simply "*i p-believes E at ω*" if $\mu(E \mid \Pi^i(\omega)) \geq p$. Denote by $B^i_p(E)$ the event "i p-believes E." That is,

$$B^i_p(E) = \{\omega\colon\ \mu(E \mid \Pi^i(\omega)) \geq p\}.$$

Notice that this is an event (i.e., measurable with respect to Σ). Moreover, it is measurable with respect to $\mathcal{F}^i$.

DEFINITION 1. An event $E \in \Sigma$ is evident p-belief if for each $i \in I$, $E \subseteq B^i_p(E)$.

DEFINITION 2. An event C is common p-belief at ω if there exists an evident p-belief event E such that $\omega \in E$, and for all $i \in I$, $E \subseteq B^i_p(C)$.

Monderer and Samet (1989) showed that the definition of common p-belief above is equivalent to the, perhaps more intuitive, following iterative definition.

DEFINITION 3. For every event C and every $0 \leq p \leq 1$, let $C_0(p) = C$, and for any $n \geq 1$, let $C_n(p) = \bigcap_{i \in I} B^i_p(C_{n-1}(p))$. The event C is common p-belief at ω if $\omega \in \bigcap_{n \geq 1} C_n(p)$.

Note that in all the above definitions, common 1-belief coincides with common knowledge up to measure 0. Lastly, we present Rubinstein's (1989) definition of "almost common knowledge."

DEFINITION 4. An event C is commonly known of degree n at ω if $\omega \in C_n(1)$.

3. SPECULATIVE TRADE AMONG RISK-NEUTRAL TRADERS

We formulate the following results for an environment similar to the one used by Milgrom and Stokey (1982). Consider an economy with a set $I = \{1, 2\}$ of traders in an uncertain environment represented by a probability space (Ω, Σ, μ), where μ is a nonatomic measure. All subsets of and functions on Ω are assumed Σ-measurable. Set inclusion should be interpreted as "μ-almost everywhere." Traders have information structures Π^i, over which they have belief operators $B_p^i(\cdot)$ for some $0 \leq p < 1$. We restrict our attention to a one-dimensional commodity space ("consumption").[2] Let $u^i\colon \mathbb{R} \to \mathbb{R}$ be trader i's increasing linear (i.e., traders are risk-neutral) von Neumann-Morgenstern utility function. Without loss of generality, we set $u^i(x) = x$ for $x \in \mathbb{R}$. Let $e^i(\omega)$ denote trader i's allocation at state ω, $e^i\colon \Omega \to \mathbb{R}$. Since the traders are risk-neutral, any pair e^1, e^2 of allocations is Pareto-efficient; that is, there does not exist a trade which is ex-ante strictly beneficial to both traders. So, without loss of generality, we set $e^1(\omega) = e^2(\omega) = 0$ for all $\omega \in \Omega$. Finally, we let $B\colon \Omega \to \mathbb{R}$ denote a proposed trade or "bet."

The following condition, dubbed *p-overlapping*, is going to play a major role in the results.

DEFINITION 5. Two information structures Π^1 and Π^2 are p-overlapping if for every trader $i \in I$, there exist a nonempty set $\pi^i = \bigcup_{k \in K^i} \Pi_k^i$ such that the following two conditions hold:

$$\mu(\pi^j \mid \Pi_k^i) \geq p \text{ for all } \Pi_k^i \subseteq \pi^i; \qquad \text{and} \tag{1}$$

$$\begin{array}{l} \text{for any two nonempty index subsets } K^{1\prime} \subseteq K^1 \\ \text{and } K^{2\prime} \subseteq K^2,\ \mu(\pi^{1\prime} \triangle \pi^{2\prime}) > 0, \text{ where} \\ \pi^{i\prime} = \bigcup_{k \in K^i} \Pi_k^i \text{ and } \triangle \text{ denotes symmetric difference.} \end{array} \tag{2}$$

p-overlapping can be interpreted as follows: (1) it is a necessary and sufficient condition for the existence of the common p-belief event $\pi^1 \cap \pi^2$; and (2) it guarantees that neither this common p-belief event, nor any of its subsets, is common 1-belief or common knowledge. The intuition behind this condition is that in order to have common p-belief of trade, the traders must agree to disagree, for otherwise, why would two risk-neutral traders wish to engage in trade in the first place? (1) is responsible for the agreement part; it guarantees the existence of an event that is the common p-belief. (2) is responsible for the disagreement part. The reason that (2) is crucial is that, as Aumann (1976) shows, agents cannot have common knowledge of disagreement.

The notion of trade involves acts, which, in turn, raise the question of rationality. It may be helpful to formulate an auxiliary result that deals only with beliefs. The following result shows that the condition of *p-overlapping*, applied to the original information partitions, characterizes the common p-belief of disagreement between traders. We introduce the following notation:

[2] We do not view this set up as conceptually restrictive. It is believed that, while it simplifies the exposition, the results follow in a more general setting as well.

let $C^i(B, v)$ denote the event where trader i, given his information, assigns expectation v to the trade B. That is, $C^i(B, v) = \{\omega\colon E(B \mid \Pi^i(\omega)) = v\}$. Define $C(B, v_1, v_2) = C^1(B, v_1) \cap C^2(B, v_2)$. $C(B, v_1, v_2)$ is the event where each trader i given his information, assigns expectation v_i to B.

THEOREM 1. *There exist a trade B and two values $v_1 \neq v_2$, such that $\mu(C(B, v_1, v_2)$ is the common p-belief) > 0, if and only if the information structures Π^1 and Π^2 are p-overlapping.*

(All proofs are relegated to the appendix.)

Let A be the set of acts available to the traders,

$$A = \{\text{“}\textit{buy},\text{” “}\textit{sell},\text{” “}\textit{refrain}\text{”}\}.$$

Let α^i denote trader i's strategy, $a^i(\omega, B, q) \in A$. That is, a^i is a function attaching to each state ω, and an offered trade B at price q, an act in A. When the offer (B, q) is implied by the context we write $a^i(\omega) \in A$ and $a(\omega) = (a^i(\omega))_{i\in I}$. Rationality is defined behaviorally as follows: a trader is rational with respect to a certain trade if, given this trade, he chooses the action that maximizes his utility, given his information and the other trader's action. Thus, rationality is actually no more than the best response of trader i to a^j, and it follows naturally that the "rationality" of one trader is defined with respect to the actions of another.[3]

Formally, we say that trader $i \in I$ is rational at $\omega \in \Omega$ with respect to a proposed trade B and a price q if $\omega \in R^i(a, B, q)$, where $R^i(a, B, q)$ is defined as

$$R^i(a, B, q) = \left\{\omega \mid a^i(\omega, B, q) \in \arg\max_{a^i \in A} E(u^i(a^i, a^j(\omega), B, q) \mid \Pi^i(\omega))\right\},$$

where $u^i(a^i(\omega), a^j(\omega), B, q)$ denotes trader i's utility given $\omega \in \Omega$, the acts (strategies) $a(\omega)$ and the fact that trade takes place if and only if one trader is willing to buy and the other to sell. That is,

$$u^i(a(\omega), B, q) = \begin{cases} u^i(e^i(\omega) + B(\omega) - q) & \text{if } a^i(\omega) = \text{“}\textit{buy},\text{” } a^j(\omega) = \text{“}\textit{sell}\text{”} \\ u^i(e^i(\omega) - B(\omega) + q) & \text{if } a^i(\omega) = \text{“}\textit{sell},\text{” } a^j(\omega) = \text{“}\textit{buy}\text{”} \\ u^i(e^i(\omega)) & \text{otherwise.} \end{cases}$$

THEOREM 2. *Let there be given pair utility functions $u^1(\cdot)$, $u^2(\cdot)$* (*not necessarily risk-neutral, or even increasing*) *and a pair of Pareto-efficient initial allocations $e^1(\cdot)$, $e^2(\cdot)$* (*not necessarily zero*). *Suppose that there exist a trade*

[3] Alternatively, one may define "rationality" independently of other traders, namely, as choosing best responses to one's beliefs. But then one would have to explicitly require that these beliefs coincide with the conditional μ. For simplicity of exposition we stick to the traditional definition of a Nash equilibrium in which beliefs are only implicit.

B, a price q, and a pair of strategies $a^1(\omega)$, $a^2(\omega)$ such that both traders are rational (*that is*, $\mu(R^i(a, B, q)) = 1$ *for* $i \in \{1, 2\}$) *and such that there is a positive probability of trade. Then, neither trader can strictly prefer trade to no-trade.*

This result, which has also been noted by Geanakopolos (1992), seems to strengthen Milgrom and Stokey's result (and even Dow, Madrigal, and Werlang's, 1990, result) significantly, since it merely requires the existence of a positive probability of trade, as opposed to common knowledge of trade. It hinges on the assumption that rational traders "know" (in an informal sense) when their partner will trade and that rationality is common knowledge at each $\omega \in \Omega$.

The immediate implication of the result above is that in our framework we must dispense with common knowledge of rationality if we want to account for trade. We therefore assume that traders act suboptimally with a probability $p > 0$ and use a weaker concept than Nash equilibrium, namely, the following.

DEFINITION 6. A pair of strategies $\{a^i(\omega, B, q)\}_{i \in I}$ constitutes a $(1 - \rho)$-rationality Nash equilibrium if for all $i \in I$, $\mu(R^i(a)) \geq 1 - \rho$. (That is, with probability at least $(1 - 2\rho)$, both traders optimize with respect to each other.)

Remark. The concept of $(1 - \rho)$-rationality Nash equilibrium is related to ex-post-ϵ-Nash equilibrium of Monderer and Samet (1989). However, while they define the deviation from Nash equilibrium or "perfect rationality" in terms of expected payoff, $(1 - \rho)$-rationality NE defines the deviation in terms of probabilities. That is, in $(1 - \rho)$-rationality NE it is required that every agent is, with probability $1 - \rho$, *perfectly* rational. Moreover, given a game, finite partitions, and an $\epsilon > 0$, there exists a $\rho > 0$ such that every $(1 - \rho)$-rationality NE is an ex-post-ϵ-Nash equilibrium. The converse, however, does not hold. An ex-post-ϵ-Nash equilibrium allows the traders to behave suboptimally for all $\omega \in \Omega$, which would correspond to a probability of irrationality $\rho = 1$. Thus, $(1 - \rho)$-rationality NE may be viewed as stricter than ex-post-ϵ-Nash equilibrium.

For simplicity we assume that irrational traders choose any suboptimal action with equal probability and that this choice is independent of the suggested trade (B, q). Note that the three actions cannot all be suboptimal, but, they may be optimal; for example, if $a^j(\omega) =$ "*refrain*" for all $\omega \in \Omega$, then $\mu(R^i(a^i, a^j)) = 1$ for all possible a^i and trades (B, q). (The suggested intuition for the case where, say, "*sell*" is the only suboptimal action, is that the trader's computer was left with a "*sell*" order that will be implemented for any offer if the trader oversleeps.) However, it should be noted that our results can be easily extended to include all the cases where a positive probability is assigned to each of the suboptimal acts on $(R^i)^c$. We do not assume that a rational trader always knows whether he is rational. This follows from the behavioral definition of rationality; that is, a trader who oversleeps may still be lucky enough to behave rationally.

We state the following result about trading in markets: fix $p < 1$ and $\rho > 0$, then, there exists a trade B and a $(1-\rho)$-rationality Nash equilibrium where the trade of B is supported by an interval of prices and is the common p-belief; ρ can be arbitrarily close to 0 and p arbitrarily close to 1. Furthermore, it is shown that the volume of trade in this case can be arbitrarily large. We introduce some more notation first; without loss of generality, let trader 1 be the buyer and trader 2 be the seller. Let $T(a, B, q)$ denote the event where both traders are rational, and trade is beneficial to both traders, that is,

$$T(a, B, q) = \left\{ \omega \in R^1 \cap R^2 \;\middle|\; \begin{array}{l} \{\text{“}\mathit{buy}\text{”}\} = \arg\max\limits_{a^1(\omega) \in A} E(u^1(a(\omega), B, q) \mid \Pi^1(\omega)) \\ \{\text{“}\mathit{sell}\text{”}\} = \arg\max\limits_{a^2(\omega) \in A} E(u^2(a(\omega), B, q) \mid \Pi^2(\omega)) \end{array} \right\}.$$

Note that by definition, “trade” implies rationality of the traders. In particular, when trade is the common p-belief, so is rationality. Also, since “*buy*” and “*sell*” are the only respective optimal actions for all $\omega \in T(a, B, q)$, the definition of trade implies that trading B at price q is strictly ex-post Pareto-improving with respect to the traders' information.[4]

THEOREM 3. *Let there be given* (*an arbitrarily small*) $\rho > 0$ *and* (*an arbitrarily large*) $p < 1$. *The following two conditions are equivalent*:

(I) *There exists a proposed trade* B, *a price* q, *and strategies* $a^1(\omega)$, $a^2(\omega)$ *such that*:

(a) *the strategies* $a^1(\omega)$, $a^2(\omega)$ *form a* $(1-\rho)$-*rationality Nash equilibrium with respect to the trade* (B, q).

(b) $\mu(T(a, B, q)$ *is the common* p-*belief*) > 0.

(II) *The information structures are* p-*overlapping*.

Furthermore, the volume of trade in this case is arbitrarily large.

Remarks. 1. Trade, as well as rationality, can also be known up to an arbitrarily large degree in a $(1-\rho)$-rationality Nash equilibrium. Specifically, given information structures that are *p-overlapping*, one may subdivide the events over which trade occurs to arbitrarily many events, in such a way that at some states in them the original events are commonly known to any prespecified degree $n \geq 1$. While n is bounded for any given information structure (say, by $2 \cdot \min_{i \in I}\{N_i\}$), and n may be supported by appropriately defined partitions.

2. While the theorem only guarantees that trade (and rationality) be common p-belief with *some* positive probability, this probability may be arbitrarily close to 1 with an appropriate choice of the information partitions. (Specifically,

[4] The term ex-post is used in this context to denote the time after ω is realized and each trader knows $\Pi^i(\omega)$. Notice that if ω itself becomes known to the traders then the traders cannot both strictly prefer trade to no trade. In fact, if $B(\omega) \neq 0$, then one of the traders would rather call the trade off.

$\mu(\pi^1 \cap \pi^2)$ may be very close to 1, or, alternatively, one may have trade over the intersection of more than one pair of $\{\pi^1, \pi^2\}$.)

3. The event of "trade" that we construct in the theorem is independent of the event "trader i is rational" given any element of trader j's partition. That is, to explain trade, one need not assume any correlation between the value of the trade and the rationality of the traders.

4. SPECULATIVE TRADE: THE GENERAL CASE

In this section we extend the results obtained in the previous section to the general case of risk averse traders. An analogous argument shows that the results hold for risk-seeking traders as well. Under risk aversion, the prices and volumes, under which trade is the common belief, reflect a "risk-premium" which is associated with the degree of risk aversion. We show that the results approximate the results obtained for risk-neutral traders as traders become less risk averse. This is hardly surprising, given that we have proved that risk-neutral traders have a strict preference for trade. On the other hand, there is some interest in this result since the volume of trade will be limited in this case.[5]

We employ the same framework of Section 3, allowing the functions u^i to reflect risk aversion. That is, preferences are represented by strictly increasing, concave, von Neumann–Morgenstern utility functions, $u^i\colon \mathbb{R} \to \mathbb{R}$. Let $e^1(\cdot)$ and $e^2(\cdot)$ denote the initial allocations that are assumed to be Pareto-efficient. The problem that arises with risk-averse traders is that they may still refuse to trade because trade increases their exposure to risk; thus, they may refrain from trade even if trade holds a positive expectation for them. What we show is that as long as traders are "sufficiently risk tolerant" they will still be willing to pay a positive price for a trade that they believe has a positive expectation.[6]

In order to proceed we need a measure of risk aversion that bypasses the difficulty associated with the Arrow–Pratt measure of risk aversion, namely, that it is only a partial ordering. The following construction provides a general framework that allows us to determine whether a trader with a given utility function can be said to be sufficiently risk tolerant for our purposes. We fix an interval $[-M, M]$. Our discussion is confined to the following family of utility

[5] Compare with Ross (1989) that states "Surely there can be nothing more embarrassing to an economist than the ability to explain the price in a market while being completely silent on the quantity."

[6] Alternatively, we can strengthen the condition of *p-overlapping* in a way that will guarantee that the constructed trade will be "noise" with respect to the traders' information and initial endowments. Consequently, the traders' decisions will be independent of their initial endowments, their willingness to pay will increase, and more risk averse traders will also be willing to engage in trade. We prefer, however, to pursue the former formulation for its simplicity and because it leads to a characterization result.

functions which is parametrized by M and ψ,

$$U_{M,\psi} = \{u\colon [-M, M] \to \mathbb{R} \mid u \text{ is concave, differentiable, strictly increasing, and } u'(M) \geq \psi\},$$

Note that $U_{\infty,0}$ is the family of general smooth preferences that exhibit risk aversion.

THEOREM 4. *Let there be given any positive ρ, M, ψ, k, and $p < 1$. The following two conditions are equivalent*:

(I) *There exists a $\delta > 0$ such that there exists a trade B with $E(B) \geq k$, a price q, and strategies $a^1(\omega)$, $a^2(\omega)$ such that for all traders with utility functions $u \in U_{M,\psi}$ that satisfy* $\sup_{\{|x|\leq M\}} |u''(x)| < \delta$ (*that is, sufficiently risk-tolerant traders*):

(a) *the strategies $a^1(\omega)$, $a^2(\omega)$ form a $(1-\rho)$-rationality Nash equilibrium with respect to the trade (B, q).*

(b) $\mu(T(a, B, q)$ *is the common p-belief*) > 0.

(II) *The information structures Π^1 and Π^2 are p-overlapping.*

As in Section 3, a similar result holds for Rubinstein's (1989) notion of almost common knowledge. Trade, as well as rationality, can also be known up to an arbitrarily large degree among sufficiently risk-tolerant traders in a $(1-\rho)$-rationality Nash equilibrium.

5. CONCLUDING REMARKS

1. In order to understand the relationship between the common p-belief and common knowledge in the context of this paper, we should check what happens as we let p approach 1. Observe that as we let p approach 1, *p-overlapping* is harder to satisfy. That is, for any given information structures that are *$\hat{p}$-overlapping* for some $\hat{p} < 1$, there exists a $\hat{p} \leq \bar{p} < 1$ such that for all $p > \bar{p}$, these information structures are not *p-overlapping*. (On the other hand, notice that for any $p \leq \hat{p}$, *p-overlapping* does hold.) Hence, given any information structure, as we let p approach 1, common p-belief of trade disappears. For $p = 1$, we are in the case of common knowledge of trade (and of rationality) and the no-trade result of Milgrom and Stokey holds. Similarly, using Rubinstein's notion of "almost common knowledge," given the Πi's, the maximal possible level of knowledge of trade is bounded by $2 \cdot \min_{i\in I}\{N_i\}$.

2. We wish to emphasize that for any $p < 1$ there exists an information structure such that for any $\rho > 0$, when preferences are fixed, there exist a bet B, a price q, and a small enough $\hat{k}$ such that for any $k \leq \hat{k}$, traders may have the common p-belief of trading $k \cdot B$. Alternatively, fix $\hat{k}$ arbitrarily large. Then, for

all sufficiently risk-tolerant preferences, traders can have the common p-belief of trade of any volume $k \leq \hat{k}$.

3. As shown by the proofs of Theorems 3 and 4, there exists a nondegenerate price interval that supports common p-belief of trade. This suggests a more active role for the "market-maker" that was mentioned in the introduction. Namely, his role would be to find a trade B that will induce the common p-belief of trade by solving a linear programming problem (as in Proposition 2 in the Appendix). Moreover, when such a trade B exists, v_1-v_2 is unbounded. Therefore the market-maker can charge a bid/ask spread and guarantee himself a positive payoff.

4. The main motivation of this paper was to explain trade in real markets, where trade is a persistent phenomenon. While our model deals only with two periods, the argument made here can be generalized in a dynamic setting to allow for repeated trade. Suppose that we have a model with infinitely many periods. Ω is fixed through time and, at period 0, $\omega \in \Omega$ is realized. The information partitions of the traders may depend on time, but at time 0, $\Pi_0^i(\omega) = \Omega$ for all $i \in I$, and traders "learn" over time so that the information partitions at time $t + 1$ are refinements of the respective information partitions at time t. At time 0, we start with Pareto-efficient allocations. In this setting, we might still get the common p-belief of trade at every period. At time t, rationality is defined as behaving optimally on that day. Trader i may want to trade, since he believes that the other traders might not be rational. After trading, the allocations may be Pareto-optimal with respect to each trader's information at time t. But, since rational behavior may be independently determined at each period and for each trade, the trader cannot be sure about the rationality of other traders and of himself in the future. Thus, at the following day, new trade might occur precisely for the same reasons. In general, arriving at Pareto-efficiency at time t might still allow for further trade at time $t + 1$ as the traders' information becomes more refined.

5. The argument of this paper can, of course, be repeated without using the common prior assumption. On the other hand, as Dow, Madrigal, and Werlang (1990) show, dispensing with the common prior assumption by itself is not enough to support trade. We should note, however, that using different priors enables us to "close" the model with each trader's prior assigning a probability $\rho > 0$ to other players being irrational, while leaving himself perfectly rational. Thus, it might be argued that using different priors alleviates some of the difficulties with the interpretation of the model, in particular regarding the way we chose to model "irrationality."

6. No-trade results are closely related to (no) agreeing-to-disagree results. Indeed, if risk-neutral traders cannot even disagree about the expected value of a certain prospect, why should they trade it if they are already in possession of a Pareto-efficient allocation? In fact, as was shown in Geanakoplos (1988) and Rubinstein and Wolinsky (1990) agreeing-to-disagree results, with the common

prior assumption, are stronger than no-trade results in the sense that they continue to hold under weaker requirements on the knowledge operator. As for relaxing these results, Monderer and Samet (1989) show that with common p-belief, the posteriors of an event cannot differ by more than $2(1-p)$.[7] So, for any $p < 1$, players can have common p-belief of disagreement over the values of the posteriors of a certain event. As belief approaches knowledge, that is, as $p \to 1$, they get the agreeing-to-disagree result.

7. The argument made here can be repeated in Aumann's (1987) setup, namely, where a state of the world specifies all objects of uncertainty, including the trader's actions. The results, in this case, will be similar to the results obtained here. Specifically, instead of postulating the existence of an equilibrium under which trade is the common p-belief, in Aumann's (1987) formulation, we can show the existence of a prior probability μ that assigns probability 1 to the event where traders adopt actions $a^1(\omega)$, $a^2(\omega)$ that form a $(1-\rho)$-rationality equilibrium and under which trade is the common p-belief.

8. The conclusion of Theorem 2 was that the assumption of common knowledge of rationality must be relaxed in order to allow for the existence of an equilibrium with trade. If, however, we replace the notion of equilibrium with the notion of rationalizability (as in Bernheim, 1984, and Pearce, 1984) we can still get trade without relaxing the assumption of common knowledge of rationality.[8] Yet, since the concept of rationalizability implicitly assumes that the traders do not know anything about other traders' choices (apart from the restrictions imposed by common knowledge of rationality), this explanation is somewhat unsatisfactory for a dynamic context. That is, over time traders are likely to gather some information about others' choices, and thus may deviate from a rationalizable n-tuple of strategies which is not an equilibrium.

APPENDIX: PROOFS

The following proposition is a characterization of the common p-belief and is a first step towards characterizing the common p-belief of trade.

PROPOSITION 1. *An event C is the common p-belief at $\omega \in \Omega$ if and only if for all $i \in I$ there exists a nonempty set $\pi^i = \bigcup_{k\in K^i} \Pi^i_k$ such that $\omega \in \pi^i$ and such that the following two conditions hold*:

$$\mu\left(\bigcap_{h\in I} \pi^h \mid \Pi^i_k\right) \geq p \qquad \text{for all } i \in I \text{ and } \Pi^i_k \subseteq \pi^i. \tag{A1}$$

$$\mu(C \mid \Pi^i_k) \geq p \qquad \text{for all } i \in I \text{ and } \Pi^i_k \subseteq \pi^i. \tag{A2}$$

[7] Neeman (1993) improves this bound to $1-p$.

[8] This fact was pointed out to me by Asher Wolinsky as well as by Roger Guesnerie.

Proof. Suppose that C is the common p-belief at $\omega \in \Omega$. There exists an evident p-belief event E such that $\omega \in E$ and $E \subseteq \bigcap_{i\in I} B_p^i(C)$. For $i \in I$, define $\pi^i = \bigcup_{\mu(\Pi_k^i \cap E)>0} \Pi_k^i$. By definition, for all $\Pi_k^i \subseteq \pi^i$ there exists an $\omega' \in E$ such that $\Pi_k^i = \Pi^i(\omega')$, and $E \subseteq \bigcap_{h\in I} \pi^h$. Therefore, it follows that $\mu(\bigcap_{h\in I} \pi^h \mid \Pi_k^i) \geq \mu(E \mid \Pi_k^i) \geq p$ for all $\Pi_k^i \subseteq \pi^i$. Similarly, $\mu(C \mid \Pi_k^i) \geq p$ for all $\Pi_k^i \subseteq \pi^i$. Conversely, define $E = \bigcap_{h\in I} \pi^h$, $\omega \in E$, so it is enough to show that E is an evident p-belief and that $E \subseteq \bigcap_{i\in I} B_p^i(C)$. Suppose that $\omega' \in E$. By definition $\Pi^i(\omega') = \Pi_k^i$ for some $\Pi_k^i \subseteq \pi^i$ and $\mu(E \mid \Pi^i(\omega')) = \mu(E \mid \Pi_k^i) \geq p$ by (A1), and $\mu(C \mid \Pi^i(\omega')) = \mu(C \mid \Pi_k^i) \geq p$ by (A2). ∎

In order to create incentives for trade it is necessary to have a proposed trade over which traders have different posterior expectations. The following proposition provides a construction of such a trade by solving a linear programming problem.

PROPOSITION 2. *Let there be given two sets $\theta^1 = \bigcup_{k=1}^n \Theta_k^1$ and $\theta^2 = \bigcup_{k=1}^m \Theta_k^2$ that are each composed of disjoint nonempty sets. There exists a trade $B\colon \Omega \to \mathbb{R}$, and $v_1 \neq v_2$ such that $E(B \mid \Theta_k^1) = v_1$ for all $\Theta_k^1 \subseteq \theta^1$, and $E(B \mid \Theta_k^2) = v_2$ for all $\Theta_k^2 \subseteq \theta^2$ if and only if θ^1 and θ^2 satisfy* (2). *Furthermore, when it exists B can be such that $E(B \mid \Theta_k^1 \backslash \theta^2) \geq v_1$ whenever it is well defined and $E(B \mid \Theta_k^2 \backslash \theta^1) \leq v_2$ whenever it is well defined.*

Proof. Define the sets $\{\Gamma_{ij}\}$ as follows: for $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, m\}$, let $\Gamma_{i,j} = \Theta_i^1 \cap \Theta_j^2$; for $j \in \{1, \ldots, m\}$, let $\Gamma_{n+1,j} = \Theta_j^2 \backslash \theta^1$; for $i \in \{1, \ldots, n\}$, let $\Gamma_{i,m+1} = \Theta_i^1 \backslash \theta^2$ and let $\Gamma_{n+1,m+1} = (\theta^1)^c \cap (\theta^2)^c$. We adopt the notation, $\gamma_{i,j} \equiv \mu(\Gamma_{i,j})$, $\lambda_i \equiv \sum_{j=1}^{m+1} \gamma_{i,j}$, and $\delta_j \equiv \sum_{i=1}^{n+1} \gamma_{i,j}$. Note that $\lambda_i, \delta_j > 0$. Consider the following linear programming problem (P),

$$
\begin{aligned}
&\text{(P)} && \max v_1 - v_2 && \\
&\text{subject to} && (\lambda_i)^{-1} \sum_{j=1}^{m+1} \gamma_{i,j} b_{i,j} = v_1 && \text{for } i \in \{1, \ldots, n\}; && \text{(A3)} \\
& && (\delta_j)^{-1} \sum_{i=1}^{n+1} \gamma_{i,j} b_{i,j} = v_2 && \text{for } j \in \{1, \ldots, m\}; && \text{(A4)} \\
& && b_{i,m+1} \geq v_1 && \text{for } i \in \{1, \ldots, n\}; \text{ and} && \text{(A5)} \\
& && b_{n+1,j} \leq v_2 && \text{for } j \in \{1, \ldots, n\}. && \text{(A6)}
\end{aligned}
$$

Notice that (P) is feasible. We write the dual problem (D) for (P).

$$
\begin{aligned}
&\text{(D)} && \min 0 \\
&\text{subject to} && (\lambda_i)^{-1}\gamma_{i,j} x_i + (\delta_j)^{-1}\gamma_{i,j} y_j = 0 \\
& && \text{for } i \in \{1, \ldots, n\};\ j \in \{1, \ldots, m\}; \qquad \text{(A7)}
\end{aligned}
$$

$$(\lambda_i)^{-1}\gamma_{i,m+1}x_i - z_i = 0$$
$$\text{for } i \in \{1,\ldots,n\}; \tag{A8}$$

$$(\delta_j)^{-1}\gamma_{n+1,j}y_j + w_j = 0$$
$$\text{for } j \in \{1,\ldots,m\}; \tag{A9}$$

$$\sum_{i=1}^{n} x_i - \sum_{i=1}^{n} z_i = -1; \tag{A10}$$

$$\sum_{j=1}^{m} y_j + \sum_{j=1}^{m} w_j = 1; \tag{A11}$$

$$\text{and where } z_i \geq 0 \text{ and } w_j \geq 0 \text{ for } i \in \{1,\ldots,n\} \text{ and } j \in \{1,\ldots,m\}, \text{ respectively.} \tag{A12}$$

We show that (P) has a feasible point with $v_1 > v_2$. This is true iff (P) is unbounded and by the duality theorem for linear programming, iff its dual (D) is infeasible. We show that the dual constraints are inconsistent.

Define the sets $\theta^{1'} = \bigcup_{x_i<0} \Theta_i^1$, $\theta^{2'} = \bigcup_{y_j>0} \Theta_j^2$.

LEMMA. *Either $\theta^{1'}$ or $\theta^{2'}$ are empty.*

Proof. Suppose otherwise. We show that in this case $\theta^{1'} \cong \theta^{2'}$ ($\cong$ denotes equal up to measure zero) in contradiction to (2). Note that $\theta^{1'} \cong \bigcup_{\gamma_{i,j}>0} \Theta_i^1$ and $\theta^{2'} \cong \bigcup_{\gamma_{i,j}>0} \Theta_i^2$. We distinguish among three cases: (1) when $i \in \{1,\ldots,n\}$ and $j \in \{1,\ldots,m\}$, (A7) implies that when $\gamma_{i,j} > 0$, $x_i < 0$ if and only if $y_j > 0$ and, therefore, $\Gamma_{i,j} \subseteq \theta^{1'}$ if and only if $\Gamma_{i,j} \subseteq \theta^{2'}$. (2) When $i \in \{1,\ldots,n\}$ and $j = m+1$, (A8) and (A12) imply that when $\gamma_{i,m+1} > 0$, $x_i \geq 0$, and, therefore, $\Gamma_{i,m+1}$ is not contained in either $\theta^{1'}$ or $\theta^{2'}$. Similarly, (3) when $i = n+1$ and $j \in \{1,\ldots,m\}$, (A9) and (A12) imply that when $\gamma_{n+1,j} > 0$, $y_j \leq 0$, and, therefore, $\Gamma_{n+1,j}$ is not contained in either $\theta^{1'}$ or $\theta^{2'}$. ■

Hence, either $x_i \geq 0$ for all $i \in \{1,\ldots,n\}$ or $y_j \leq 0$ for all $j \in \{1,\ldots,m\}$. In the former case it follows that $x_i \geq z_i \geq 0$ for all $i \in \{1,\ldots,n\}$ and $\sum_{i=1}^n x_i - \sum_{i=1}^n z_i \geq 0$, in contradiction to (A10); and in the latter, it follows that $y_j \leq -w_j \leq 0$ for all $j \in \{1,\ldots,m\}$ and $\sum_{i=1}^m y_j + \sum_{i=1}^m w_j \leq 0$ in contradiction to (A11). Therefore, we conclude that (D) is infeasible.

To prove the other direction, suppose there exist nonempty index subsets $K^{1'} \subseteq \{1,\ldots,n\}$ and $K^{2'} \subseteq \{1,\ldots,m\}$ such that $\mu(\theta^{1'}\Delta\theta^{2'}) = 0$. It follows that $v_1 = E(B \mid \theta^{1'}) = E(B \mid \theta^{2'}) = v_2$, in contradiction to $v_1 \neq v_2$. ■

Proof of Theorem 1. By Proposition 1, the fact that $C(B, v_1, v_2)$ is the common p-belief implies the existence of sets $\pi^1 = \bigcup_{k\in K^1} \Pi_k^1$ and $\pi^2 = \bigcup_{k\in K^2} \Pi_k^2$ such that $\mu(\pi^1 \cap \pi^2 \mid \Pi_k^i) \geq p$ and $\mu(C(B, v_1, v_2) \mid \Pi_k^i) \geq p$ for $i \in \{1, 2\}$ and all $\Pi_k^i \subseteq \pi^i$ which implies (1). π^1 and π^2 satisfy (2) as well. Otherwise, there exist nonempty sets $\pi^{i\prime} = \bigcup_{k\in K^{i\prime}} \Pi_k^i$ for $i \in I$ such that $\mu(\pi^{1\prime}\Delta\pi^{2\prime}) = 0$. But, $\pi^{1\prime} \cap \pi^{2\prime} \subseteq B_p^i(C(B, v_1, v_2))$; therefore for $\omega \in \pi^{1\prime} \cap \pi^{2\prime}$, $E(B \mid \Pi^1(\omega)) = v_1$ and $E(B \mid \Pi^2(\omega)) = v_2$. However, since $\mu(\pi^{1\prime}\Delta\pi^{2\prime}) = 0$, $v_1 = E(B \mid \bigcup_{k\in K^{1\prime}} \Pi_k^1) = E(B \mid \bigcup_{k\in K^{2\prime}} \Pi_k^2) = v_2$, which contradicts $v_1 \neq v_2$. Conversely, suppose that Π^1 and Π^2 are p-overlapping. There exist two nonempty sets $\pi^1 = \bigcup_{k\in K^1} \Pi_k^1$ and $\pi^2 = \bigcup_{k\in K^2} \Pi_k^2$ that satisfy (1) and (2) and by Proposition 2 there exist a bet B and $v_1 \neq v_2$, such that $E(B \mid \Pi_k^1) = v_1$ for all $\Pi_k^1 \subseteq \pi^1$, and $E(B \mid \Pi_k^2) = v_2$ for all $\Pi_k^2 \subseteq \pi^2$. By Proposition 1, to show that $C(B, v_1, v_2)$ is common p-belief for all $\omega \in \pi^1 \cap \pi^2$ we need only to show that $\mu(C(B, v_1, v_2) \mid \Pi_k^i) \geq p$ for all $i \in I$ and $\Pi_k^i \subseteq \pi^i$. This holds since $\pi^i \subseteq C^i(B, v_i)$ for $i \in I$. ∎

Proof of Theorem 2. The conditions imply that there exists a positive probability event $T = \{\omega \mid a^i(\omega, B, q) = \text{“}buy\text{”}, a^j(\omega, B, q) = \text{“}sell\text{”}\}$, where trader i buys B at price q, and trader j sells it. The rationality of the traders implies that for any $\omega \in \Omega$,

$$E(u^i(e^i(\omega) + (B(\omega) - q) \cdot 1_T) \mid \Pi^i(\omega)) \geq E(u^i(e^i(\omega)) \mid \Pi^i(\omega))$$

aggregating over the different $\Pi^i(\omega)$'s, we get

$$E(u^i(e^i(\omega) + (B(\omega) - q) \cdot 1_T)) \geq E(u^i(e^i(\omega)))$$

Similarly, for trader j,

$$E(u^j(e^j(\omega) - (B(\omega) - q) \cdot 1_T)) \geq E(u^j(e^j(\omega))).$$

Thus, if even one of the traders strictly benefits from trade, we obtain a contradiction to the Pareto-optimality of the initial allocations e^i and e^j. ∎

Proof of Theorem 3. (I) implies (II) Suppose that there exists a proposed trade B, a price q, and strategies a^1, a^2 such that $T(a, B, q)$ is common p-belief at $\omega \in \Omega$. By Proposition 1 there exist two sets π^1 and π^2 that satisfy (A1) and (A2). (A1) coincides with (1). For any $\omega \in T(a, B, q)$, “*buy*” is the unique optimizing action on $\Pi^1(\omega)$, and since trader 1 is rational in $T(a, B, q)$, $a^1(\omega') =$ “*buy*” for all $\omega' \in \Pi^1(\omega)$ as well. Similarly, $a^2(\omega') =$ “*sell*” for all $\omega' \in \Pi^2(\omega)$ such that $\omega \in T(a, B, q)$. Suppose now that (2) fails to hold. There exist, then, nonempty index subsets $K^{1\prime} \subseteq K^1$ and $K^{2\prime} \subseteq K^2$ such that $\mu(\pi^{1\prime}\Delta\pi^{2\prime}) = 0$. It follows that at $\omega \in \pi^{1\prime} \cap \pi^{2\prime}$, $\pi^{1\prime} \cap \pi^{2\prime}$ is common knowledge, and so at $\omega \in \pi^{1\prime} \cap \pi^{2\prime}$, there is common knowledge of strictly improving trade among rational traders which contradicts Theorem 2.

(II) implies (I) Suppose that $\pi^1 = \bigcup_{k\in K^1} \Pi_k^1$ and $\pi^2 = \bigcup_{k\in K^2} \Pi_k^2$ satisfy (1) and (2). Define the sets $\{\Theta_k^1\}_{k\in K^1}$ and $\{\Theta_k^2\}_{k\in K^2}$ as follows: for $i, j \in I$,

$i \neq j, k \in K^i$ such that $\Pi_k^i \subseteq \pi^j$, let $\Theta_k^i = \Pi_k^i$. Otherwise, for $i, j \in I, i \neq j$, $k \in K^i$ such that $\Pi_k^i \not\subset \pi^j$, choose $F_k^j \subseteq \Pi_k^i \backslash \pi^j$ with $0 < \mu(F_k^j) \leq \rho/N_i$, and let $\Theta_k^i = (\Pi_k^i \cap \pi^j) \cup F_k^j$. $F^j = \bigcup_{k \in K^i} F_k^j$ will be the event over which trader j is irrational. It is straightforward to verify that $\theta^1 = \bigcup_{k \in K^1} \Theta_k^1$ and $\theta^2 = \bigcup_{k \in K^2} \Theta_k^2$ satisfy (2). By Proposition 2 there exists a trade B such that $E(\mathrm{B} \mid \Theta_k^1) = v_1$ for all $\Theta_k^1 \subseteq \theta^1$, $E(\mathrm{B} \mid \Theta_k^2) = v_2$ for all $\Theta_k^2 \subseteq \theta^2$, $v_2 < v_1$, and where $\mathrm{E}(B \mid F_k^1) \leq v_2$ and $E(B \mid F_k^2) \leq v_1$ whenever $\mu(F_k^1) > 0$ and $\mu(F_k^2) > 0$, respectively. Without loss of generality, B can be assumed to be a constant over all sets of the form $\Pi_k^1 \cap \Pi_h^2$. Fix $q \in (v_2, v_1)$ and define the strategies a^1 and a^2 as follows:

$$a^1(\omega, B, q) = \begin{cases} \text{“}buy\text{”} & \omega \in \pi^1 \cup F^1 \\ \text{“}refrain\text{”} & \text{otherwise;} \end{cases}$$

$$a^2(\omega, B, q) = \begin{cases} \text{“}sell\text{”} & \omega \in \pi^2 \cup F^2 \\ \text{“}refrain\text{”} & \text{otherwise.} \end{cases}$$

We show that a forms a $(1-\rho)$-rationality NE and that $T(a, B, q)$ is held as common p-belief at any $\omega \in \pi^1 \cap \pi^2$. For any $\omega \in \Pi^1(\omega) \subseteq \pi^2$, “*buy*” is the optimal action for trader 1; because B's conditional expectation is v_1, it is offered at a price $q < v_1$, and trade takes place with probability 1. For $\omega \in \Pi^1(\omega) \not\subseteq \pi^2$ such that $\Pi^1(\omega) \cap \pi^2 \neq \varnothing$, “*buy*” is still the only optimal action for trader 1, because by construction B's expected value conditional on $\Pi^1(\omega)$ and on the event of trade is still v_1. For $\omega \notin \pi^1 \cup \pi^2$, trader 2 refrains from trade and so any action is optimal. Finally, for $\omega \in \pi^2 \backslash \pi^1$ $E(B \mid \Pi^1(\omega) \cap \{\omega \mid a^2(\omega, B, q) =$ “*sell*”$\}) \leq v_2$. Therefore, buying B at a price $q > v_2$ is irrational. Thus trader 1 is irrational at $\omega \in F^1$ and rational otherwise. A similar argument shows that trader 2 is rational everywhere except F^2. By construction $\mu(F^i) \leq \rho$ and it therefore follows that $a(\omega B, q)$ is a $(1-\rho)$-rationality NE equilibrium. To complete the proof note that $T(a, B, q)$ is held as the common p-belief at any $\omega \in \pi^1 \cap \pi^2$ because it satisfies (A2) and π^1 and π^2 satisfy (A1). Finally, the risk-neutrality of the traders implies that they are willing to trade arbitrarily large quantities of B. ■

Before presenting the proof of Theorem 4, we present the following lemma.

LEMMA. *Let there be given two random variables X and Y such that $|X(\omega)|$, $|Y(\omega)| \leq M/2$ for all $\omega \in \Omega$, and two positive constants ψ and c. Let T and Π be two measurable subsets of Ω such that $\mu(T \cap \Pi) > 0$ and $E(Y \mid T \cap \Pi) \geq 0$. Then,*

(i) *There exists a $\delta > 0$ such that for all $u \in U_{M,\psi}$ satisfying* $\sup_{\{|x| \leq M\}} |u''(x)| < \delta$, $E(u(X + (Y + c) \cdot 1_T) \mid \Pi) > E(u(X) \mid \Pi)$, *and*;

(ii) *Define $\hat{c}(u)$ such that $E(u(X + (Y + \hat{c}(u)) \cdot 1_T) \mid \Pi) = E(u(X) \mid \Pi)$. For all $\epsilon > 0$ there exists a $\delta > 0$ such that for all $u \in U_{M,\psi}$ satisfying* $\sup_{\{|x| \leq M\}} |u''(x)| < \delta$, $\hat{c}(u) < \epsilon$.

Proof. (i) We record the following fact. Its proof is straightforward and is omitted.

FACT. *For all* $\epsilon > 0$ *there exists a* $\delta > 0$ *such that for all* $u \in U_{\infty,0}$ *satisfying* $\sup_{\{|x|\leq M\}} |u''(x)| < \delta$ *there exists a linear function* v: $\mathbb{R} \to \mathbb{R}$, *such that* $v' \geq u'(M)$ *and* $\sup_{\{|x|\leq M\}} |u(x) - v(x)| < \epsilon$.

Set $\epsilon < (\psi \cdot c/2)\mu(T \cap \Pi)$. By the fact above, there exists a $\delta > 0$ such that for all $u \in U_{M,\psi}$ satisfying $\sup_{\{|x|\leq M\}} |u''(x)| < \delta$, there exists a linear function v such that $v' \geq u'(M)$ and $v(x) - \epsilon < u(x) < v(x) + \epsilon$ for all $|x| \leq M$. Therefore,

$$\begin{aligned} E(u(X + (Y + c) \cdot 1_T) \mid \Pi) &> E(v(X + (Y + c) \cdot 1_T) \mid \Pi) - \epsilon \\ &\geq E(v(X) \mid \Pi) + \psi \cdot c - \epsilon \\ &> E(v(X) \mid \Pi) + \epsilon \\ &> E(u(X) \mid \Pi). \end{aligned}$$

(ii) Fix an $\epsilon > 0$, by (i) there exists a $\delta > 0$ such that for all $u \in U_{M,\psi}$ satisfying $\sup_{\{|x|\leq M\}} |u''(x)| < \delta$, $E(u(X + Y + \epsilon) \mid T) > E(u(X) \mid T)$. Replacing ϵ with $\hat{c}(u)$ causes the last inequality to become an equality; therefore we deduce $\hat{c}(u) < \epsilon$. ∎

Proof of Theorem 4. (I) implies (II) The proof is identical to the proof of Theorem 3.

(II) implies (I) The proof follows the proof of Theorem 3 while making the necessary corrections for risk. As in Theorem 3, we generate the sets θ^1 and θ^2 and a trade B that satisfies all the conditions specified in the proof of Theorem 3. Without loss of generality, we multiply B such that $E(B) \geq k$. Define $\hat{c}^i(\Pi_k^i)$ such that $E(u^i(e^i(\omega) + (-1)^{i-1}(B - \hat{c}^i(\Pi_k^i))) \mid \Pi_k^i) = E(u^i(e^i(\omega)) \mid \Pi_k^i)$ for all $\Pi_k^i \subseteq \pi^i$ and denote $\hat{c}^i = \min_{\{\Pi_k^i \subseteq \pi^i\}}\{\hat{c}^i(\Pi_k^i)\}$. By the previous lemma there exist a $\delta > 0$ such that if traders' utilities satisfy $\sup_{\{|x|\leq M\}} |u''(x)| < \delta$ then there exists a price $q \in (v_2 + \hat{c}^2, v_1 - \hat{c}^1)$. Consider the strategies a^1 and a^2 that were defined in the proof of Theorem 3. These strategies constitute a $(1 - \rho)$-rationality NE. As in the proof of Theorem 3, on π^1 trader 1 is rational; when he is also sufficiently risk tolerant, "*buy*" is the unique optimal action when the trade B is proposed at the price q. On F^1 trader 1 is irrational and "*buy*" is the unique suboptimal action. On $\Omega \backslash (\pi^1 \cup F^1)$, "*refrain*" is an optimal action. Similarly, trader 2 is rational everywhere except on F^2. Hence, a^1 and a^2 constitute a $(1 - \rho)$-rationality NE for sufficiently risk-tolerant traders. Finally, as in the proof of Theorem 3, $T(a, B, q)$ is held as the common p-belief at any $\omega \in \pi^1 \cap \pi^2$. ∎

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