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The effectiveness of English auctions

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Abstract

We study the performance of the English auction under different assumptions about the seller's degree of "Bayesian sophistication." We define the effectiveness of an auction as the ratio between the expected revenue it generates for the seller and the expected valuation of the object to the bidder with the highest valuation (total surplus). We identify tight lower bounds on the effectiveness of the English auction for general private-values environments, and for private-values environments where bidders' valuations are non-negatively correlated. For example, when the seller faces 12 bidders who the seller believes have non-negatively correlated valuations whose expectations are at least as high as 60% of the maximal possible valuation, an English auction with no reserve price generates an expected price that is more than 80% of the value of the object to the bidder with the highest valuation.

JEL classification: D44; D82

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1. Introduction

The notion of optimality is central to economics. Yet, economic theory typically only distinguishes between optimal and sub-optimal outcomes. By and large, there is no attempt to quantify how far from optimality are sub-optimal outcomes, allocations, or institutions.

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In this paper we attempt to make a preliminary step toward quantification of optimality in the context of auction theory. Specifically, we study the performance of the (single-object) English auction in private-values environments.[3] We define the *effectiveness* of an auction as the ratio between the expected revenue it generates for the seller and the expected valuation of the object to the bidder with the highest valuation (which coincides with total surplus when bidders are risk-neutral or risk-averse).[4] We identify tight lower bounds on the effectiveness of the English auction for several "classes" of environments and under three different assumptions about the seller's degree of "Bayesian sophistication" as expressed in his ability to set an appropriate reserve price. Specifically, we consider the English auction with no reserve price, with a fixed positive reserve price and with an optimally chosen reserve price. Our results show that the English auction performs reasonably well in a wide class of environments. As will become clearer below, they may be interpreted as quantifying the "optimality" of the English auction, or alternatively, as establishing the "cost" of relying on the "simple" English auction relative to the "optimal" auction in those circumstances where the latter extracts the full surplus.

At a perhaps more practical level, in recent years, several governments around the world have auctioned off parts of the electromagnetic (airwave) spectrum for commercial use with the main stated objective of promoting efficiency or "putting licenses into the hands of those who value them most"[5] (Milgrom, 1996, Chapter 1, p. 3). While the revenues obtained exceeded expectations by a factor of ten or more, to the extent that maximizing revenue is also an important objective, as it is likely to be in private auctions, existing theory provides no way of assessing the effectiveness of the auction form used in terms of what fraction of the total sum of bidders' willingness to pay was obtained. The method described in this paper provides a first step towards being able to form such assessments.

Generally, the effectiveness of any auction form depends on the environment that is considered. In any particular environment, the closer effectiveness is to one, the closer the auction is to extracting the full surplus. We seek to determine the effectiveness of the English auction in the environment, within a given class of possible environments, in which it is the lowest. In this sense, we perform *worst-case* analysis of the performance of the English auction. Our results illustrate the robustness of the English auction in the following

[3] The English auction has many variants. One such variant that is referred to by Milgrom and Weber (1982) as the Japanese version of the English auction is idealized as follows. Before the auction begins, the bidders are given the opportunity to inspect the object and realize their valuations. The bidders choose whether to be active at the start price that is equal to the reserve price set by the seller. As the auctioneer raises the price, bidders drop out one by one. No bidder who has dropped out can become active again. The auction ends as soon as no more than one bidder remains active. The remaining bidder gets the object for the prevailing price. If several bidders dropped out simultaneously, ending the auction, one of these bidders is chosen randomly and gets the object at the price at which she quit.

In private values environments, the English auction is outcome-equivalent to a modified Vickrey auction, or a sealed-bid second-price auction where the seller may set a reserve price (see, e.g., (Milgrom and Weber, 1982)). All of our results therefore apply to second price auctions as well.

[4] In the natural sciences, concepts that quantify aspects of the quality of performance are widely used. For example, the notion of "energetic efficiency" which is defined as "what you get out of some device divided by what you put in" (Vogel, 1998, p. 156) is similar to the notion of effectiveness presented here.

[5] For details, see McAfee and McMillan (1996) and Milgrom (1996).

sense: A seller who is uncertain about which environment he is facing within a certain class of environments is guaranteed an expected revenue (as a proportion of total surplus) that is not lower and in general higher than worst-case effectiveness. Thus, for those classes of environments for which the worst-case effectiveness of the English auction can be shown to be "high," a seller who is uncertain about the environment, is unable to figure out the optimal auction, and even if he is able, is suspicious about whether bidders understand the optimal auction's rules and is doubtful whether they employ Bayesian–Nash equilibrium strategies, is well advised to employ an English auction; even in the worst-case, his losses from not doing otherwise will be small.

We establish the following results. We parametrize all possible private-values environments by the number of bidders, n, and their expected valuations of the object, α, as a percent of the maximum possible valuation. We obtain a lower bound on the ratio between the expected revenue generated by the English auction and expected total surplus under three different assumptions on the seller's behavior:

(1) the seller does not set a reserve price,
(2) the seller sets a fixed reserve price that maximizes his expected revenue given his beliefs about the *expectations* of bidders' valuations for the object, and
(3) the seller sets an optimal reserve price given his belief about the *distribution* of bidders' valuations for the object.

These three assumptions can be thought of as corresponding to three different levels of "Bayesian sophistication." From a seller that fails to recognize the fact that setting a positive reserve price may increase his expected revenue, to a seller who recognizes the usefulness of a reserve price but is unable to articulate a belief about bidders' valuations beyond a specification of their expected valuations, to a seller who can fully articulate his beliefs and set an optimal reserve price accordingly.[6] Some readers have pointed to an apparent tension between our assumption that the seller may be so rational so as to be capable of setting an optimal reserve price given his beliefs, and our method of analysis which focuses on a "low rationality" worst-case analysis. We believe that in light of the fact that even for the simple environments considered in this paper (correlated private values environments with risk averse-bidders), the problem of identifying the optimal auction is still very much an open one,[7] the decision to employ an English auction with an optimally chosen reserve price, especially when the worst-case performance of this auction is reasonable, is very sensible. Moreover, the fact that for private-values environments with non-negatively correlated bidders' valuations the differences between the worst-case performance of the English auction with and without a reserve price is quite

[6] We ignore the issue of whether the seller's beliefs are "correct." In a Bayesian world, beliefs are subjective. The best that any Bayesian rational agent can ever do is to maximize with respect to her beliefs.

[7] Even for correlated general values environments with risk neutral bidders, where optimal auctions that succeed in extracting the entire bidders' surplus have been identified (Crémer and McLean, 1988; McAfee and Reny, 1992), the optimality of these auctions depends on the controversial assumption that the seller's and the bidders' beliefs are consistent. An assumption that is not needed here. See Neeman (1999) for additional discussion of this and related points.

small, suggests that less sophisticated sellers should find employing English auctions (even without a reserve price) even more sensible.

For each level of the seller's degree of Bayesian sophistication and every given n and α, we determine the effectiveness of the English auction in the *worst possible* private-values environment. The identified bounds are tight, and we present examples of environments that attain them. We then repeat this exercise for environments where bidders' valuations are non-negatively correlated. We assume, specifically, that bidders' valuations of the object are conditionally independent and identically distributed.

As expected, the worst-case effectiveness of the English auction improves as the number of bidders, n, and their expected valuations for the object, α, increase. Obviously, when bidders' valuations are negatively correlated, the possibility of setting an appropriately chosen reserve price is very valuable. For example, consider an environment with two bidders who have negatively correlated valuations such that when one bidder's valuation is one, the other bidder's valuation is zero and vice versa. In such an environment, an English auction with no reserve price generates an expected revenue of zero, but an English auction with a reserve price of one, generates an expected revenue that is equal to the total surplus, one. The English auction is more effective when bidders' valuations are more plausibly assumed to be non-negatively correlated. For example, when $n = 12$ and $\alpha = 60\%$, even in the worst possible case, an English auction with no reserve price sells the object at an expected price that is more than 80% of the expected value of the object to the bidder with the highest valuation.

The research that is most closely related to the work reported here is the series of papers that culminated in the work of Rustichini et al. (1994) (see also the discussion in Section 6 below). Rustichini et al. demonstrated that the inefficiency of double-auctions under symmetric equilibria in i.i.d. environments converges to zero at an asymptotic rate of the order of magnitude of $c/(nm)$ where n is the number of buyers, m is the number of sellers, and c is a constant that depends on the particular environment considered. We perform a similar exercise on the English auction. However, instead of considering efficiency, we focus on seller's revenues and consider a much wider class of environments without restricting the set of equilibria. Furthermore, whereas Rustichini et al. only identified asymptotic rates of convergence, we identify tight lower bounds for any number of bidders.[8] More recently, Satterthwaite and Williams (1999) have shown that the double-auction is also worst-case asymptotically optimal. That is, there does not exist any other exchange mechanism that has a faster asymptotic rate of convergence to efficiency.

The rest of the paper is organized as follows. In the next section we present the model and state the general problem. We present the results for general private-values environments in Section 3, and the results for private-values environments where bidders' valuations are non-negatively correlated in Section 4. By focusing on worst rather than, say, "average" performance, worst-case analysis tends to emphasize the "weakness" rather than "strength" of a mechanism. We therefore devote Section 5 to analysis of the worst-case effectiveness of another commonly used sale mechanism—the posted-price mechanism.

[8] Recently, Swinkels (1998, 1999) established the asymptotic efficiency of discriminatory auctions and a class of uniform price auctions for multiple identical goods in private values environment where bidders' valuations are independent but there may be some aggregate uncertainty about demand and supply.

The results compare unfavorably with those of the English auction. The point of the comparison is not to argue that from the seller's perspective the English auction is superior to the posted-price mechanism, we believe that much is obvious, but rather to "calibrate" the readers' expectations about what constitutes "reasonable" worst-case performance. We conclude in Section 6 with an additional discussion of motivation and related literature. All proofs and a short explanation about our calculations are relegated to Appendix A.

2. The model

We consider general private-values environments. A seller has a single object to sell. There are n potential buyers (bidders) for the object. The object is worth nothing for the seller but has a value $v_i \in [0,1]$ for bidder $i \in \{1,\dots,n\}$. The payoff to bidder i from buying the object at a price p depends only on her valuation for the object v_i and the price paid. We assume that it is given by $u_i(v_i - p)$ where bidder i's payoff function, $u_i : \mathbb{R} \to \mathbb{R}$, is assumed to be increasing and such that $u_i(0) = 0$. The payoff to the bidder when she does not buy the object (and does not pay) is normalized to zero. We assume that each bidder knows her valuation for the object. Except for that, we make no other assumption about the bidders' beliefs. In particular, we do not assume that a common prior exists, nor that the buyers' and seller's beliefs are consistent.

If the seller had complete information about the bidders' valuations for the object and full bargaining power, he could fetch an expected price of

$$R_{FB} = E\bigl[\max\{v_1,\dots,v_n\}\bigr]$$

for the object by selling it at the price $\max\{v_1,\dots,v_n\}$ to the bidder with the highest valuation. Since the seller's valuation for the object is zero, if in addition we assume that the buyers are risk-neutral or risk-averse, then R_{FB} describes the maximal expected total surplus that can be generated by selling the object.

We employ the following notation: we let $B : [0,1]^n \to [0,1]$ denote the environment facing the seller, or more precisely, the beliefs that the seller would have had about the environment he is facing had he been a sophisticated Bayesian. We let $G : [0,1] \to [0,1]$ denote the corresponding cumulative distribution of $\max\{v_1,\dots,v_n\}$ and $H : [0,1] \to [0,1]$ denote the corresponding cumulative distribution of the second highest valuation from among $\{v_1,\dots,v_n\}$. By a well-known equality (see, e.g., (Shiryaev, 1989, p. 208))

$$R_{FB}(B) = \int_0^1 \bigl(1 - G(x)\bigr)\,\mathrm{d}x. \tag{1}$$

We are interested in obtaining a lower bound on the ratio between the expected revenue that the seller can obtain for the object when he employs an English auction and the maximal total surplus as defined above in (1).[9] The expected revenue to the seller depends

[9] We do not *assume* that the seller is risk-neutral, however, a very risk-averse seller would obviously not care much for our results.

on whether he sets a reserve price for the object or not. We consider three different types of seller's behavior that correspond to the three different degrees of the seller's Bayesian sophistication as described above: (1) whether it is because the seller cannot articulate any beliefs about the buyers' valuations for the object or because the seller does not realize that setting a positive reserve price may increase the expected revenue generated by the auction, the seller does not set a reserve price. (2) The seller recognizes the benefit conferred by setting a positive reserve price, but because he cannot articulate beliefs about buyers' valuations that are more specific than the buyers' expected valuations, he sets a reserve price given his limited beliefs. A lower bound on the seller's expected revenue in this case is given by the seller adopting a maxmin approach, namely, choosing a reserve price that maximizes the expected revenue for the seller in the environment in which it is the lowest. Finally, (3) the seller sets an optimal reserve price given his beliefs about the distribution of the bidders' valuations.

Denote the expected revenue to the seller from employing an English auction with a reserve price r in the environment B by $R_E(B,r)$. Given a vector of bidders' valuations $v_1,\dots,v_n$, let $x_1,\dots,x_n$ denote the ordered vector of bidders' valuations where x_1 denotes the largest valuation from among $v_1,\dots,v_n$, x_2 denotes the second largest valuation from among $v_1,\dots,v_n,\dots,$ and x_n denotes the smallest valuation from among $v_1,\dots,v_n$. In English auctions, it is a dominant strategy for the bidders to remain active in the auction until the price equals their valuations for the object. It therefore follows that

$$R_E(B,r) = r\Pr(x_1 \geqslant r) + \Pr(x_2 \geqslant r)E[x_2 - r|x_2 \geqslant r].$$

That is, the seller obtains a revenue of r whenever the highest valuation of the bidders, x_1, is larger or equal to r and an additional revenue of $x_2 - r$ when the second highest valuation is also larger or equal to r. Integration by parts yields,

$$R_E(B,r) = r\bigl(1 - G(r^-)\bigr) + \int_r^1 \bigl(1 - H(x)\bigr)\,\mathrm{d}x \tag{2}$$

where $G(r^-) = \lim_{x\nearrow r} G(x)$.

Define the *effectiveness* of the English auction with reserve price r in the environment B by

$$\frac{R_E(B,r)}{R_{FB}(B)}.$$

Note that what appears in the denominator of the definition of effectiveness is the total surplus generated by the sale rather than the expected revenue the seller can obtain by employing an optimal auction. For the class of environments where the bidders are risk neutral, have correlated valuations, and hold beliefs that are consistent with those of the seller's, Crémer and McLean (1988) and McAfee and Reny (1992) showed that the optimal auction succeeds in extracting the entire bidders' surplus. However, for the type of environments considered here, where the bidders may be risk-averse, and consistency is not assumed, a characterization of the optimal auction is unavailable. To the extent that in these more general cases the optimal auction falls short of extracting the entire buyers' surplus, our measure of effectiveness under-estimates the effectiveness of the English auction relative to that of the "optimal" auction.[10]

As explained above, for a fixed class of environments $\mathcal{B}$, we are interested in identifying a lower bound on the effectiveness of the English auction under three different assumptions about the seller's behavior. Specifically, we ask what is

$$\mathcal{E}_{\mathcal{B}}^{0} \equiv \min_{B\in\mathcal{B}} \left\{ \frac{R_E(B,0)}{R_{FB}(B)} \right\}, \tag{3}$$

or the effectiveness of the English auction with no reserve price in the environment $B \in \mathcal{B}$ in which it is the lowest;

$$\mathcal{E}_{\mathcal{B}}^{r(n,\alpha)} \equiv \max_{r\in[0,1]} \left\{ \min_{B\in\mathcal{B}} \left\{ \frac{R_E(B,r)}{R_{FB}(B)} \right\} \right\}, \tag{4}$$

or the effectiveness of the English auction with a positive reserve price that cannot be tailored to suit the specific environment the seller faces, in the environment $B \in \mathcal{B}$ in which it is the lowest; and

$$\mathcal{E}_{\mathcal{B}}^{r(B)} \equiv \min_{B\in\mathcal{B}} \left\{ \max_{r(B)\in[0,1]} \left\{ \frac{R_E(B,r(B))}{R_{FB}(B)} \right\} \right\}, \tag{5}$$

or the effectiveness of the English auction with a reserve price $r(B)$ that is chosen optimally given the seller's beliefs about the buyers' valuations B, in the environment $B \in \mathcal{B}$ in which it is the lowest. Thus, for example, a seller that employs an English auction with no reserve price is guaranteed an expected revenue that is at least as high as $\min_{B\in\mathcal{B}}\{R_E(B,0)/R_{FB}(B)\}$ of the expected total surplus when facing any environment in the class $\mathcal{B}$.

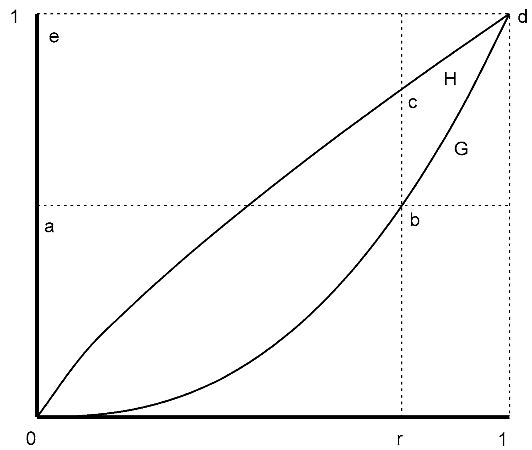

Fig. 1. G, H, R_{FB}, and $R_E(r)$.

[10] The English auction with an optimally chosen reserve price is optimal in the class of i.i.d. private-values environments with risk-neutral bidders (Myerson, 1981), hence its effectiveness relative to the optimal auction in such environments is one.

Our method of proof makes extensive use of geometric arguments and intuition. The effectiveness of an English auction with reserve price r for seller's beliefs that induce distributions G and H of the first and second highest valuations from among $v_1, \ldots, v_n$, is represented in Fig. 1 by the ratio between the areas abcde (which by (2) is equal to $R_E(B,r) = r(1 - G(r^-)) + \int_r^1 (1 - H(x))\,\mathrm{d}x$) and 0bde (which by (1) is equal to $R_{FB}(B) = \int_0^1 (1 - G(x))\,\mathrm{d}x$).

The problem of determining the worst-case environment and worst-case effectiveness is equivalent to the problem of identifying seller's beliefs that induce a minimal ratio of the areas abcde and 0bde.

3. General private values environments

Let $\mathcal{B}_{n,\alpha}$, $\alpha \in [0,1]$, denote the set of cumulative joint distribution functions of n random variables that obtain their values on the unit interval and have expectations larger or equal to α. We interpret $\mathcal{B}_{n,\alpha}$ as representing the general class of private-values environments with n bidders whose expected valuations are larger than or equal to α% of the maximal possible valuation. The parameter α describes how high, on average, bidders' valuations are. It may be interpreted as describing the "attractiveness" of the underlying auction environment as perceived by the seller.

Denote the worst-case effectiveness of the English auction with no reserve price, with a fixed reserve price that cannot be tailored to the specific environment, and with a reserve price that is chosen optimally given the seller's beliefs in the class of environments $\mathcal{B}_{n,\alpha}$ by $\mathcal{E}_n^0(\alpha)$, $\mathcal{E}_n^{r(n,\alpha)}(\alpha)$, and $\mathcal{E}_n^{r(B)}(\alpha)$, respectively.

Theorem 1. *For every $n \geqslant 2$ and $\alpha \in (0,1]$,*

$$\mathcal{E}_n^0(\alpha) = \max\left\{\frac{n\alpha - 1}{n-1}, 0\right\}. \tag{6}$$

Thinking of the bidders' valuations as n random variables $v_1, \ldots, v_n$ with expectation larger or equal to α, the idea of the proof is to identify the joint distribution that induces the highest possible expectation of the first order statistic from among $v_1, \ldots, v_n$, but the smallest possible expected second order statistic. The proof shows that the environment that attains the worst-case bound is one where one bidder, which the seller believes is equally likely to be any one of the bidders, has valuation 1 with probability $\min\{n\alpha, 1\}$ and valuation 0 otherwise, and all other bidders have valuations $\max\{(n\alpha - 1)/(n-1), 0\}$. Note that according to these beliefs, bidders' valuations are negatively correlated.

For the case where the seller cannot tailor the reserve price to the specific environment, we have,

Theorem 2. *For every $n \geqslant 2$ and $\alpha \in (0,1]$,*

$$\mathcal{E}_n^{r(n,\alpha)}(\alpha) = \begin{cases} \dfrac{r(n,\alpha)n(\alpha - r(n,\alpha))}{(1-r(n,\alpha))(n\alpha-(n-1)r(n,\alpha))}, & 0 \leqslant \alpha \leqslant \dfrac{1+\sqrt{4n-3}}{2n}, \\ \dfrac{n\alpha-1}{n-1}, & \dfrac{1+\sqrt{4n-3}}{2n} \leqslant \alpha \leqslant 1, \end{cases} \tag{7}$$

where

$$r(n,\alpha) = \frac{\alpha(n-\sqrt{(n-1)\alpha})}{n-1+\alpha} \quad \text{when } 0 \leqslant \alpha \leqslant \frac{1+\sqrt{4n-3}}{2n}, \tag{8}$$

and

$$r(n,\alpha) \in \left[0, \frac{n\alpha-1}{n-1}\right] \quad \text{when } \frac{1+\sqrt{4n-3}}{2n} \leqslant \alpha \leqslant 1. \tag{9}$$

The idea of the proof is to identify for every possible reserve price $r \in [0,1]$, the joint distribution that minimizes the effectiveness of the English auction for this particular r, and then to maximize over r. When $0 \leqslant \alpha \leqslant (1+\sqrt{4n-3})/(2n)$, for every reserve price $r < \alpha$,[11] the distribution that attains worst-case effectiveness is one where with the highest possible probability given the constraint that bidders' expected valuations must be larger or equal to α, the bidder with the highest valuation has a valuation that is either equal to 1 or just below the reserve price r, and all other bidders have valuations just below the reserve price r. Maximization of this lowest possible effectiveness over r reveals that the distribution that attains the worst case bound is one where one bidder, which the seller believes is equally likely to be any one of the bidders, has valuation 1 with probability $n(\alpha - r(n,\alpha))/(1-r(n,\alpha))$ and valuation just below $r(n,\alpha)$ otherwise, and all other bidders have valuations just below $r(n,\alpha)$. When $(1+\sqrt{4n-3})/(2n) \leqslant \alpha \leqslant 1$, it is impossible to ensure that all the bidders' valuations except the highest one are below $r \leqslant (n\alpha-1)/(n-1) < \alpha$, and so worst-case effectiveness is identical to that obtained when the seller is constrained to set the reserve price equal to zero. In this case, the ability to set a positive reserve price does not help the seller. The ability to set a fixed positive reserve price thus helps the seller only when $\alpha \leqslant (1+\sqrt{4n-3})/(2n)$, or when the bidders' expected valuations are small relative to their number. Intuitively, a fixed positive reserve price helps the seller when n and α are low.

For the case where the seller chooses the reserve price optimally given his beliefs, we have,

Theorem 3. *For every $n \geqslant 2$ and $\alpha \in (0,1]$,*

$$\mathcal{E}_n^{r(B_n)}(\alpha) = \frac{n\alpha - \beta}{(n-1)\beta} \tag{10}$$

where β is the unique solution in the interval $[0,1]$ to the equation:

$$\left(\frac{n\alpha-\beta}{n-1}\right)\left(1-\log\left(\frac{n\alpha-\beta}{n-1}\right)\right) = \beta. \tag{11}$$

[11] It is straightforward to verify that for $r \geqslant \alpha$ worst-case effectiveness is equal to zero.

The idea of the proof is that in the distribution that attains the lowest bound, it must be that the seller is indifferent between setting an optimal reserve price for the bidder with the highest valuation while ignoring all other bidders, and not setting any reserve price at all. Otherwise, as the proof shows, it is possible to change the distribution and decrease effectiveness. The distribution that attains the worst case bound is one where one bidder, which the seller believes is equally likely to be any one of the bidders, has a valuation that is distributed according to a truncated Pareto distribution,

$$F_\varepsilon(x) = \begin{cases} 0, & 0 \leqslant x < \varepsilon, \\ 1 - \varepsilon/x, & \varepsilon \leqslant x < 1, \\ 1, & x = 1, \end{cases} \tag{12}$$

where $\varepsilon \in [0, 1]$ is such that $\varepsilon(1 - \frac{1}{n}\log(\varepsilon)) = \alpha$, and all the other bidders have valuations equal to ε. Observe that, first, the seller believes that every bidder's expected valuation of the object is equal to α. Second, more importantly, again, the distribution that attains the worst-case bound is one where the bidders' valuations are negatively correlated. Finally, third, under the distribution that attains the worst-case bound, every reserve price set by the seller generates the same expected revenue ε for the seller. Obviously, any reserve price $r \leqslant \varepsilon$ yields a revenue equal to the second highest valuation ε. As for reserve prices $r \in (\varepsilon, 1]$, with probability $F_\varepsilon(r)$, the seller does not sell the object (and obtains a revenue of 0), and with probability $1 - F_\varepsilon(r)$, the seller succeeds in selling the object for the price r. The seller's expected revenue is therefore given by $r(1 - F_\varepsilon(r))$, which can be immediately seen to equal ε for every $r \in [\varepsilon, 1]$. Moreover, F_ε is the only function that satisfies the property that,

$$r\bigl(1 - F_\varepsilon(r)\bigr) = \varepsilon \quad \text{for every } r \in [\varepsilon, 1]. \tag{13}$$

We depict the values of $\mathcal{E}_n^0(\alpha)$, $\mathcal{E}_n^{r(n,\alpha)}(\alpha)$, and $\mathcal{E}_n^{r(B)}(\alpha)$, for $n = 4, 12$, and the limits as n tends to infinity, in Table 1.[12]

Table 1
Worst-case effectiveness for general environments

α	$\mathcal{E}_4^0(\alpha)$	$\mathcal{E}_4^{r(4,\alpha)}(\alpha)$	$\mathcal{E}_4^{r(B)}(\alpha)$	$\mathcal{E}_{12}^0(\alpha)$	$\mathcal{E}_{12}^{r(12,\alpha)}(\alpha)$	$\mathcal{E}_{12}^{r(B)}(\alpha)$	$\mathcal{E}_\infty^0(\alpha), \mathcal{E}_\infty^{r(\infty,\alpha)}(\alpha)$	$\mathcal{E}_\infty^{r(B)}(\alpha)$
0	0	0	0	0	0	0	0	0
.1	0	.047649	.2605	.01818	.06531	.28644	.1	.30279
.2	0	.10286	.33136	.12727	.14288	.36427	.2	.38322
.3	.0667	.16791	.39566	.23636	.2369	.43339	.3	.45373
.4	.2	.24621	.46007	.34545	.34545	.501	.4	.52184
.5	.33333	.34315	.52768	.45455	.45455	.5701	.5	.59062
.6	.46667	.46667	.60078	.56364	.56364	.64262	.6	.66189
.7	.6	.6	.6816	.67273	.67273	.72018	.7	.7371
.8	.73333	.73333	.77274	.78182	.78182	.80437	.8	.81757
.9	.86667	.86667	.87743	.89091	.89091	.89695	.9	.90468
1	1	1	1	1	1	1	1	1

[12] The limits can be shown to be equal to $\lim_{n\to\infty} \mathcal{E}_n^0(\alpha) = \lim_{n\to\infty} \mathcal{E}_n^{r(n,\alpha)}(\alpha) = \alpha$, and $\lim_{n\to\infty} \mathcal{E}_n^{r(B_n)}(\alpha) = 1/(1 - \log(\alpha))$, respectively. The difference between these limits and the values of $\mathcal{E}^0$, $\mathcal{E}^{r(\alpha)}$, and $\mathcal{E}^{r(B_n)}$ when $n = 25$ is already quite small. For $\mathcal{E}_{25}^0 = \mathcal{E}_{25}^{r(25,\alpha)}$, where it is the largest, it is smaller than .04.

Note that worst-case effectiveness stays bounded away from 1 as the number of bidders tends to infinity even when the seller sets the reserve price optimally given his beliefs. This apparently counter-intuitive result is due to the fact that on the distributions that attain the worst-case bounds, the bidders' valuations are negatively correlated. Each of the three distributions that attain the worst-case bounds above describes the case of a seller who believes that he faces n bidders out of which only one, which he cannot identify, is "serious" and is willing to pay a high price for the object while all the others are no more than "warm bodies" with relatively low valuations. Even when the seller sets the reserve price optimally, and can identify the serious bidder, the fact that the serious bidder's valuation is distributed according to the truncated Pareto distribution that satisfies the special property (13) implies that the seller cannot get a high expected revenue from this bidder. It should not come as a surprise that when there is only one serious bidder, increasing the number of bidders has a negligible effect on the effectiveness of the English auction.

4. Private-values environments with non-negatively correlated bidders' valuations

While the general class of environments, or more precisely, the general class of seller's beliefs about the environment, is the appropriate class in some applications, in many other cases the appropriate class of environments is smaller. In particular, given that assuming that bidders' valuations are non-negatively correlated accords well with the qualitative features of real life auctions (for this reason, it is also the maintained assumption in much of auction literature), we restrict our attention in this section to this case.

Specifically, let $\mathcal{B}_{n,\alpha}^{ciid}$ denote the set of cumulative distribution functions that describe the joint distribution of n random variables that obtain their values on the unit interval, have expectations larger or equal to α, and are conditionally independent and identically distributed (c.i.i.d.). The set $\mathcal{B}_{n,\alpha}^{ciid}$ describes the beliefs of sellers who believe that there is some unobservable factor that affects all bidders' valuations in the same way. For example, bidders' valuations may depend on the (unobservable to the seller) state of the economy, or on the "intrinsic worth of the object," or on both. Any such (possibly multi-dimensional) unobservable factor introduces positive correlation into the distribution of bidders' valuations but, conditional on it, the bidders' valuations are independently and identically distributed.[13,14]

[13] By de Finetti's theorem (see, e.g., (Durrett, 1991, p. 232)) an infinite sequence of random variables is conditionally i.i.d. if and only if it is exchangeable. Diaconis and Freedman (1980) describe a sense in which a finite sequence of exchangeable random variables is approximately conditionally i.i.d.

[14] Another widely used assumption that implies non-negative correlation in auction models is affiliation (Milgrom and Weber, 1982). While many examples of distributions of bidders' valuations are both conditionally i.i.d and affiliated, the two notions are independent. Examples of conditionally i.i.d random variables that are not affiliated are easy to construct; for an example of affiliated random variables that are not conditionally i.i.d., see (Shaked, 1979, p. 72 (ii)).

Thus, every environment $B \in \mathcal{B}_{n,\alpha}^{ciid}$ may be represented as a mixture of i.i.d. distributions in the following way,

$$B(v_1, \ldots, v_n) = \int \prod_{i=1}^{n} F(v_i \mid z)\, \mathrm{d}Z(z) \tag{14}$$

where for every $z \in \mathcal{Z}$, $F(\cdot \mid z) \in \mathcal{B}_{1,\alpha(z)}$ for some $\alpha(z) \in (0, 1]$ and $\int \alpha(z)\, \mathrm{d}Z(z) \geqslant \alpha$.

Denote the worst-case effectiveness of the English auction with no reserve price, with a reserve price that cannot be tailored to the specific environment, and with a reserve price that is chosen optimally given the seller's beliefs in the class of conditionally i.i.d. environments $\mathcal{B}_{n,\alpha}^{ciid}$ by $\mathcal{E}_n^{0,ciid}(\alpha)$, $\mathcal{E}_n^{r(n,\alpha),ciid}(\alpha)$, and $\mathcal{E}_n^{r(B),ciid}(\alpha)$, respectively. We have the following result.

Theorem 4. *For every $\alpha \in (0, 1]$, $\mathcal{E}_n^{0,ciid}(\alpha)$ is obtained on joint distributions of the form $B(v_1, \ldots, v_n) = \int \prod_{i=1}^{n} F(v_i \mid z)\, \mathrm{d}Z(z) \in \mathcal{B}_{n,\alpha}^{ciid}$ where for every z, $F(\cdot \mid z) \in \mathcal{B}_{1,\alpha(z)}$ is a "two-step" distribution function of the form*:

$$F(x) = \begin{cases} 0, & x \in (-\infty, 0), \\ p, & x \in [0, b), \\ q, & x \in [b, 1), \\ 1, & x \in [1, \infty), \end{cases} \tag{15}$$

for some $0 \leqslant p \leqslant q \leqslant 1$ and $0 \leqslant b \leqslant 1$, and $\mathcal{E}_n^{r(n,\alpha),ciid}(\alpha)$ is obtained on a limit of a sequence of such distributions.

The idea of the proof is to show that unless every $F(\cdot \mid z)$ is a two-step function with support on at most three points 0, b, and 1, the environment B can be changed so as to decrease effectiveness. In fact, we conjecture, a conjecture that is confirmed by our numerical analysis but which we cannot prove, that worst-case effectiveness is obtained on a mixture of two-step distribution functions where the "first step" is always equal to 0 (i.e., $p = 0$ in (15) above from which it follows that the distribution is supported by only two points, b and 1). Intuitively, when the reserve price is constrained to be zero, conditional on z, under such distributions the expectation of the highest of the bidders' valuation is "maximized," whereas the expectation of the second highest of the bidders' valuations is "minimized." Remarkably, for the case where the reserve price is constrained to be zero, our numerical analysis reveals that worst-case effectiveness is obtained on degenerate mixtures of two-step distribution functions, namely, on i.i.d. distributions of two-step functions. Again, we conjecture that this is generally the case, but we cannot prove it. Even remarkably still, our numerical analysis also reveals that the ability to set a fixed positive reserve price does not allow the seller to obtain higher worst-case effectiveness, $\mathcal{E}_n^{0,ciid}(\alpha)$ seems to be equal to $\mathcal{E}_n^{r(n,\alpha),ciid}(\alpha)$ for every value of $n \geqslant 2$ and $\alpha \in (0, 1]$.

We depict the computed values of $\mathcal{E}_n^{0,ciid}(\alpha) = \mathcal{E}_n^{r(n,\alpha),ciid}(\alpha)$ for $n = 4, 12$, and 25 in Table 2 (see Appendix A for a short explanation on how the calculation was performed).

As suggested by Table 2, even when the seller sets no reserve price, worst-case effectiveness converges to 1 for every $\alpha \in (0, 1]$.

Table 2
Worst-case effectiveness for conditionally i.i.d. environments

α	$\mathcal{E}_4^{0,ciid}(\alpha), \mathcal{E}_4^{r(4,\alpha),ciid}$	$\mathcal{E}_{12}^{0,ciid}(\alpha), \mathcal{E}_{12}^{r(12,\alpha),ciid}$	$\mathcal{E}_{25}^{0,ciid}(\alpha), \mathcal{E}_{25}^{r(25,\alpha),ciid}$
0	0	0	0
.1	.15208	.32411	.36106
.2	.30600	.47259	.50317
.3	.43872	.57934	.60472
.4	.54811	.66507	.68591
.5	.64273	.73769	.75442
.6	.72696	.80117	.81413
.7	.80337	.85785	.86729
.8	.87362	.90925	.91537
.9	.93889	.95638	.95937
1	1	1	1

Table 3
Upper bounds on worst-case effectiveness with optimally chosen reserve price in i.i.d. environments

α	$\mathcal{E}_4^{r(B),ciid}(\alpha) \leqslant$	$\mathcal{E}_{12}^{r(B),ciid}(\alpha) \leqslant$	$\mathcal{E}_{25}^{r(B),ciid}(\alpha) \leqslant$
0	0	0	0
.1	.31415	.45226	.5883
.2	.41496	.56441	.65178
.3	.50583	.65517	.70938
.4	.59351	.73015	.76190
.5	.67969	.79224	.80998
.6	.75400	.84005	.85416
.7	.82260	.88439	.89489
.8	.88612	.92563	.93255
.9	.94509	.96407	.96750
1	1	1	1

Unfortunately, we cannot analytically identify the distributions that attain worst case effectiveness for c.i.i.d. environments where the seller sets the reserve price optimally given his beliefs. The fact that, for example, in Table 2, on the environment in $\mathcal{B}_{n,\alpha}^{ciid}$ that attains worst case effectiveness with no reserve price for $n = 12$ and $\alpha = 10\%$, choosing the reserve price optimally increases effectiveness from .32411 to .89579, but for $n = 12$ and $\alpha = 50\%$, choosing the reserve price optimally increases effectiveness only from .73769 to .75282 suggests that setting the reserve price optimally may significantly improve worst-case effectiveness for environments with low α's, but it may have only a negligible effect on worst-case effectiveness for environments with larger α's.

However, we are able to determine the worst-case effectiveness of the English auction with an optimally chosen reserve price for i.i.d. environments that satisfy some regularity condition as shown in Table 3 for values of $n = 4$, 12, and 25.[15]

[15] The proof is quite involved and is not reproduced here. It can be obtained from the author upon request.

The fact that for α larger than 40%, the differences between the values depicted in Tables 2 and 3 are below 10% demonstrates that, unless α is small, an optimally chosen reserve price does not improve worst-case effectiveness by much on conditionally i.i.d. environments.[16]

5. Comparison to the posted price mechanism

In this section we determine the worst-case effectiveness of the posted price mechanism. As explained in the introduction, the motivation for this exercise is not to argue for the superiority of the English auction over posted-prices, but rather to "calibrate" expectations as to what constitutes "reasonable" worst-case performance.

As Milgrom (1989, p. 18) writes "Posted prices are commonly used for standardized, inexpensive items sold in stores." Several authors examined the relative performance of posted-prices compared to auctions (see (Wang, 1993; Kultti, 1999) and the references therein) and compared to bargaining (see (Wang, 1995) and the references therein) and described conditions under which the posted-price mechanism may outperform, or at least perform as well, as either auctions or bargaining.

As in the rest of the literature, we assume that posted prices are set optimally given the seller's beliefs about the distribution of the buyers' valuations for the object. We have the following result,

Theorem 5. *For every $n \geqslant 2$ and $\alpha \in (0, 1]$, the worst-case effectiveness of the posted-price mechanism is given by*

$$\mathcal{E}^{PP}(\alpha) = \frac{1}{1 - \log(\varepsilon)} \tag{16}$$

where ε is the unique solution to $\varepsilon(1 - \log(\varepsilon)) = \alpha$ in the interval $(0, 1]$. It is obtained on the distribution (in $\mathcal{B}^{ciid}_{n,\alpha}$) where all the buyers have the same identical valuation that is distributed according to the truncated Pareto distribution F_ε in (12).

Since English auctions with optimally chosen reserve prices obviously dominate reserve prices alone which are equivalent to the posted-price mechanism, the more interesting comparison is between the English auction with no reserve price and the posted-price mechanism. These two sale mechanisms perform better in very different types of environments. The English auction with no reserve price performs relatively well when bidders' valuations are positively correlated, and relatively poorly when bidders' valuations are negatively correlated. In contrast, the posted-price mechanism performs relatively well

[16] For larger α's, the fact that the possibility of setting an optimal reserve price is not very valuable for the seller is consistent with Bulow and Klemperer's (1996) result that an English auction with no reserve price and $n + 1$ bidders generates a higher expected revenue than an English auction with an optimally chosen reserve price but n bidders. However, it should be emphasized that the two-step environments that attain worst case effectiveness violate one of the conditions (specifically, downward-sloping MR) that is maintained throughout Bulow and Klemperer's analysis.

Table 4
Worst-case effectiveness of the posted-price mechanism

α	0	.1	.2	.3	.4	.5	.6	.7	.8	.9	1
$\mathcal{E}^{PP}(\alpha)$	0	.20451	.25036	.29076	.33088	.37336	.42080	.47679	.54813	.65282	1

when buyers' valuations are negatively correlated because by charging a high price, the seller can extract more surplus from the buyer with the highest valuation, but it performs relatively poorly in environments where the buyers' valuations are positively correlated because it cannot exploit the implied "competition" among the buyers.

We depict the value of $\mathcal{E}^{PP}(\alpha)$, which is independent of the number of buyers, n, in Table 4.

In spite of the fact that, especially when we restrict our attention to c.i.i.d. environments, the worst-case performance of the English auction is better than that of the posted-price mechanism, the worst-case performance of the latter compares favorably with that of the former for low n's and α's.

6. Discussion

The inspiration for this paper came from what has been called the "Wilson critique." Wilson emphasized that in contrast to optimal mechanisms that are tailored to specific environments, the rules of real economic institutions "are not changed as the environment changes; rather they persist as stable, viable institutions" (1987, p. 36). In Wilson (1985), he argued that good economic institutions must not rely on features that are common knowledge among the agents such as (in the context of auctions) the number of potential bidders, the bidders' and seller's probability assessments (i.e., the prior), and the functional form of the dependence of the bidders' willingness to pay for the object on their types. While asking that mechanisms be independent of whatever is commonly known among the agents seems somewhat extreme, it is upheld by the fact that in "practical situations," little, if at all, is commonly known among the relevant agents. Wilson (1985) presented the double-auction as a premier example of a simple institution, (obviously, the English auction provides another such example), and demonstrated its Pareto incentive efficiency when the number of buyers and sellers is large. We described some of the research that has followed in the introduction.

Except for the literature on double-auctions mentioned above, the literature that is most closely related to our work in its motivation is the one that identifies environments in which simple mechanisms are optimal. The motivating idea is that if it can be demonstrated that these environments are general enough, then the prevalence of simple mechanisms is explained.[17] In auction theory, the early work of Vickrey (1961) showed that the most widely used auction forms such as the English, the Dutch, and the sealed-bid first-price auctions are equivalent in terms of the expected revenues they generate for the

[17] McAfee (1992, p. 284), for example, writes "Finding the restriction that leads to the optimality of simple mechanisms ... [is] the most important problem facing mechanism design."

seller in private values environments in which the bidders' valuations are independently and identically distributed. Myerson (1981) then established the optimality of these auction forms in these environments.[18] More recently, Lopomo (1998) showed that an "augmented" English auction (where the seller sets the reserve price optimally after all but one of the bidders dropped out) maximizes the expected revenue for the seller among all ex-post incentive compatible and ex-post individually rational auction mechanisms. In contract theory, Holmstrom and Milgrom (1987), Laffont and Tirole (1987), and McAfee and McMillan (1987) have established the optimality of linear incentive contracts in specific classes of environments. In contrast to this literature that demonstrates the optimality of simple mechanisms in special environments in order to explain their prevalence in general environments, the approach taken here is to focus on one well-known simple auction mechanisms—the English auction—and to show that while perhaps strictly sub-optimal in most environments, it is nevertheless reasonably effective in a wide range of plausible environments.

Finally, as for the merit of worst-case analysis: if there existed an agreed upon prior over the set of all possible environments, a better indicator of the effectiveness of the English auction could be given by its average (according to this prior) rather than its worst-case performance. The fact that such a prior does not exist indicates that the nature of the uncertainty facing the seller is difficult to quantify in terms of (objective) risk. Under such circumstances, worst-case analysis still allows us to form sensible judgements about the quality of performance of the English auction in spite of the fact that discussion of "average performance" is impossible.

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Appendix A. Proofs

We begin by identifying $\mathcal{E}_1^{r(1,\alpha)}(\alpha)$ and $\mathcal{E}_1^{r(B_1)}(\alpha)$ (it is straightforward to see that $\mathcal{E}_1^0(\alpha)=0$ for all $\alpha \in (0,1]$). Auctions with only one bidder are not very interesting, but the results illustrate our method of proof and will become useful later.

Lemma 1. *For every $\alpha \in (0,1]$,*

$$\mathcal{E}_1^{r(1,\alpha)}(\alpha)=\frac{r(\alpha-r)}{(1-r)\alpha} \quad \textit{where } r=1-\sqrt{1-\alpha}.$$

[18] However, Myerson (1981) followed by Crémer and McLean (1985, 1988) and McAfee and Reny (1992) also showed that, even within the confines of the private values model, when the identical distribution and then the independence assumptions are relaxed, the resulting optimal auctions are very different from any auction that is used in practice.

Proof. Fix an $\alpha \in (0,1]$ and a reserve price $r \in [0,1)$. We show that (1) the worst case effectiveness of the English auction with reserve price r over the environments in $\mathcal{B}_{1,\alpha}$ is larger or equal to $r(\alpha - r)/((1-r)\alpha)$; and (2) describe a family of distributions $\{B_\varepsilon^\alpha\}_{\varepsilon>0} \subseteq \mathcal{B}_{1,\alpha}$ such that for every $\varepsilon > 0$, the effectiveness of the English auction with reserve price r in the environment B_ε^α is equal to $r(\alpha - r + \varepsilon)/((1-r+\varepsilon)\alpha) \searrow r(\alpha - r)/((1-r)\alpha)$, when $\varepsilon \searrow 0$. Finally, (3) we show that the highest worse case effectiveness over the environments in $\mathcal{B}_{1,\alpha}$ is obtained at the reserve price $r = 1 - \sqrt{1-\alpha}$.

(1) Consider any cumulative distribution function B that induces an expectation α. Since

$$\alpha = \int_0^1 \bigl(1 - B(x)\bigr)\,\mathrm{d}x = \int_0^r \bigl(1 - B(x)\bigr)\,\mathrm{d}x + \int_r^1 \bigl(1 - B(x)\bigr)\,\mathrm{d}x \leqslant r + (1-r)\bigl(1 - B(r^-)\bigr),$$

where $B(r^-) = \lim_{r' \nearrow r} B(r)$, it must be that $B(r^-) \leqslant (1-\alpha)/(1-r)$. Therefore,

$$R_E(B,r) = r\bigl(1 - B(r^-)\bigr) \geqslant r\left(1 - \frac{1-\alpha}{1-r}\right) = \frac{r(\alpha - r)}{1-r}.$$

Now, since $r(\alpha - r)/((1-r)\alpha)$ is increasing in α, for every $\alpha \in (0,1]$ the worst case effectiveness of the English auction with reserve price r over the environments in $\mathcal{B}_{1,\alpha}$ is larger or equal to $r(\alpha - r)/((1-r)\alpha)$.

(2) For every $\alpha \in (0,1]$, $r \in [0,\alpha)$ and $\varepsilon > 0$, define the cumulative distribution function $B_\varepsilon^{\alpha,r} \in \mathcal{B}_{1,\alpha}$ as follows: $B_\varepsilon^{\alpha,r}(x) = 0$ for $x \in [0, r-\varepsilon)$, $B_\varepsilon^{\alpha,r}(x) = (1-\alpha)/(1-r+\varepsilon)$ for $x \in [r-\varepsilon, 1)$ and $B_\varepsilon^{\alpha,r}(1) = 1$ (suppose that $\varepsilon < r$ so that $B_\varepsilon^{\alpha,r}$ is indeed a cumulative distribution function). It is straightforward to verify that the effectiveness of the English auction with reserve price r in the environment $B_\varepsilon^{\alpha,r}$ is $r(\alpha - r + \varepsilon)/((1-r+\varepsilon)\alpha) \searrow r(\alpha - r)/((1-r)\alpha)$, when $\varepsilon \searrow 0$.

(3) Finally, for a given $\alpha \in (0,1)$, we compute the reserve price r that generates the highest worse case effectiveness over $\mathcal{B}_{1,\alpha}$. (When $\alpha = 1$, $r = 1$ is the best reserve price.) That is, we compute $\arg\max_{r\in[0,1]}\{r(\alpha - r)/((1-r)\alpha)\}$. Since $\frac{\mathrm{d}}{\mathrm{d}r}(r(\alpha - r)/((1-r)\alpha)) = (r^2 - 2r + \alpha)/((1-r)^2\alpha)$, the maximum is obtained at the smaller root of $r^2 - 2r + \alpha$, namely $r = 1 - \sqrt{1-\alpha}$. □

Lemma 2. *For every $\alpha \in (0,1]$,*

$$\mathcal{E}_1^{r(B)}(\alpha) = \frac{1}{1 - \log(\varepsilon)} = \frac{\varepsilon}{\alpha}$$

where ε is the unique solution to $\varepsilon(1 - \log(\varepsilon)) = \alpha$ in the interval $(0,1]$.

Proof. Fix an $\alpha \in (0,1]$. Consider any distribution function $B \in \mathcal{B}_{1,\alpha}$ that is not a truncated Pareto distribution F_ε as in (12) where $\varepsilon \in [0,1]$ is such that $\varepsilon(1 - \log(\varepsilon)) = \alpha$. The fact that B is different from F_ε and that both belong to $\mathcal{B}_{1,\alpha}$, i.e., $\int_0^1 (1 - B(x))\,\mathrm{d}x \geqslant \int_0^1 (1 - F_\varepsilon(x))\,\mathrm{d}x = \alpha$ implies that there must exist an $\hat{x} \in (\varepsilon, 1]$ such that $B(\hat{x}) < F_\varepsilon(\hat{x})$. Therefore,

$$\frac{\max_{r\in[0,1]}\{R_E(B,r)\}}{\alpha} \geqslant \frac{R_E(B,\hat{x})}{\alpha} = \frac{\hat{x}(1 - B(\hat{x}))}{\alpha} > \frac{\hat{x}(1 - F_\varepsilon(\hat{x}))}{\alpha} = \frac{\varepsilon}{\alpha}$$

where the last equality follows from (13). □

The proofs of Theorems 1–3 proceed in two steps. First, in Lemma 3 below we show that among all the distributions $B \in \mathcal{B}_{n,\alpha}$ that induce the same distribution of the highest valuation G, the minimal ratios $\mathcal{E}_n^0(\alpha; x_1 \sim G)$, $\mathcal{E}_n^{r(n,\alpha)}(\alpha; x_1 \sim G)$, and $\mathcal{E}_n^{r(B)}(\alpha; x_1 \sim G)$, are obtained at the distribution that induces the distribution of the second highest valuation H^* that is described below. Then, we show that among all the distributions $B \in \mathcal{B}_{n,\alpha}$ that induce a distribution of the second highest valuation H^*, the minimal ratios $\mathcal{E}_n^0(\alpha; x_2 \sim H^*)$, $\mathcal{E}_n^{r(n,\alpha)}(\alpha; x_2 \sim H^*)$, and $\mathcal{E}_n^{r(B)}(\alpha; x_2 \sim H^*)$ are obtained on certain distributions of the highest valuation that are described below.

Lemma 3. *Among all the distributions $B \in \mathcal{B}_{n,\alpha}$ that induce a distribution of the highest valuation G, the minimal ratios $\mathcal{E}_n^0(\alpha; x_1 \sim G)$, and $\mathcal{E}_n^{r(B)}(\alpha; x_1 \sim G)$, are obtained on those distributions that induce a distribution of the second highest valuation that is given by*

$$H^*(x) = \begin{cases} G(x), & \text{for } 0 \leqslant x < \hat{x}, \\ 1, & \text{for } \hat{x} \leqslant x \leqslant 1, \end{cases} \tag{17}$$

where $\hat{x}$ is such that $\int_0^1 (1 - H^*(x))\,\mathrm{d}x = (n\alpha - \beta)/(n-1)$ and $\beta = \int_0^1 (1 - G(x))\,\mathrm{d}x$.

Proof. The proof formalizes the intuition that for any given reserve price r and distribution of the highest valuation G, worst-case effectiveness is obtained on those distributions where the second highest valuation is as small as possible (or, where H is "pushed" to the left as much as possible). Fix a reserve price $r \in [0,1]$ and a distribution $B \in \mathcal{B}_{n,\alpha}$ that induces a distribution of the highest valuation G such that $\int_0^1 (1 - G(x))\,\mathrm{d}x = R_{FB}(B) = \beta$. Denote the distribution of the second highest valuation by H.

We show that $\int_r^1 (1 - H(x))\,\mathrm{d}x \geqslant \int_r^1 (1 - H^*(x))\,\mathrm{d}x$. This follows immediately for $r \geqslant \hat{x}$. We assume therefore that $r < \hat{x}$. Recall that $x_1, \ldots, x_n$ denote the largest to smallest valuations from among $v_1, \ldots, v_n$, where the latter are distributed according to B. Because $\sum_{i=1}^n E[x_i] = \sum_{i=1}^n E[v_i] \geqslant n\alpha$ and $E[x_1] = \beta$, $E[x_2] + \sum_{i=3}^n E[x_i] \geqslant n\alpha - \beta$, and since $E[x_2] \geqslant E[x_i]$ for all $i \in \{3, \ldots, n\}$,

$$E[x_2] \geqslant \frac{n\alpha - \beta}{n-1}. \tag{18}$$

Since $R_E(B_n, 0)/R_{FB}(B_n) = E[x_2]/\beta$, the proof for $\mathcal{E}_n^0$ where $r = 0$ ends here. Suppose that $\int_r^1 (1 - H(x))\,\mathrm{d}x < \int_r^1 (1 - H^*(x))\,\mathrm{d}x$ where $r < \hat{x}$. Because H is the distribution of x_2, $H(x) \geqslant G(x) = H^*(x)$ for all $x \in [0, r]$. Therefore,

$$\begin{aligned} E[x_2] &= \int_0^r \big(1 - H(x)\big)\,\mathrm{d}x + \int_r^1 \big(1 - H(x)\big)\,\mathrm{d}x < \int_0^r \big(1 - G(x)\big)\,\mathrm{d}x + \int_r^1 \big(1 - H^*(x)\big)\,\mathrm{d}x \\ &= \int_0^1 \big(1 - H^*(x)\big)\,\mathrm{d}x = \frac{n\alpha - \beta}{n-1}. \end{aligned}$$

A contradiction to (18). Therefore, for every $r \in [0,1]$ and distribution $B \in \mathcal{B}_{n,\alpha}$ that induces a distribution of highest valuation G such that $\int_0^1 (1 - G(x))\,\mathrm{d}x = \beta$,

$$\frac{R_E(B, r)}{R_{FB}(B)} = \frac{r(1 - G(r^-)) + \int_r^1 (1 - H(x))\,\mathrm{d}x}{\beta} \geqslant \frac{r(1 - G(r^-)) + \int_r^1 (1 - H^*(x))\,\mathrm{d}x}{\beta}.$$

The last inequality holds for every $r \in [0,1]$, and in particular, for the r that maximizes the last expression. Therefore, among all the distributions $B \in \mathcal{B}_{n,\alpha}$ that induce the distribution of the highest valuation G, the minimal ratios $\mathcal{E}_n^{r(n,\alpha)}(\alpha; x_1 \sim G)$ and $\mathcal{E}_n^{r(B)}(\alpha; x_1 \sim G)$ are obtained at the distribution that induces a distribution of the second highest valuation that is given by H^*. □

Proof of Theorem 1. By Lemma 3, we may restrict our attention to those distributions $B \in B_{n,\alpha}$ that induce a distribution of the second highest valuation that is given by H^*. We show that among all such distributions, the minimal ratio $E_n^0(\alpha; x_2 \sim H^*) = \max\{(n\alpha - 1)/(n-1), 0\}$ is obtained on a distribution where one (randomly chosen) bidder has valuation 1 with probability $\min\{n\alpha, 1\}$ and valuation 0 otherwise, and all other bidders have valuations $\max\{(n\alpha - 1)/(n-1), 0\}$.

Fix a distribution $B \in \mathcal{B}_{n,\alpha}$ that induces a distribution of the highest valuation G and a distribution of the second highest valuation H^*. Recall that $E[x_1] = \beta$. By definition of H^*,

$$E[x_2] = \frac{n\alpha - \beta}{n-1}.$$

Therefore, minimal effectiveness is equal to,

$$\min_{\beta\in[0,1]}\left\{\frac{E[x_2]}{E[x_1]}\right\}=\max\left\{\min_{\beta\in[0,1]}\left\{\frac{n\alpha-\beta}{(n-1)\beta}\right\},0\right\}=\max\left\{\frac{n\alpha-1}{n-1},0\right\}$$

since $(n\alpha-\beta)/((n-1)\beta)$ is decreasing in β. $\square$

Proof of Theorem 2. Fix an $n\geqslant 2$ and $\alpha\in(0,1)$. By Lemma 3, we may restrict our attention to those distributions $B\in B_{n,\alpha}$ that induce a distribution of the second highest valuation that is given by H^*. We show that worst-case effectiveness among such distributions is given by

$$\mathcal{E}_n^{r(n,\alpha)}(\alpha;x_2\sim H^*)=\begin{cases}\dfrac{r(n,\alpha)n(\alpha-r(n,\alpha))}{(1-r(n,\alpha))(n\alpha-(n-1)r(n,\alpha))}, & \alpha^2n(1-\alpha)\leqslant 1,\\ \dfrac{n\alpha-1}{n-1}, & \alpha^2n(1-\alpha)>1,\end{cases}\tag{19}$$

where $r(n,\alpha)=\alpha(n-\sqrt{n-n\alpha})/(n-1+\alpha)$ when $\alpha^2n(1-\alpha)\leqslant 1$. When $\alpha^2n(1-\alpha)>1$, every reserve price in the interval $[0,(n\alpha-1)/(n-1)]$ is optimal. When $\alpha^2n(1-\alpha)\leqslant 1$, worst-case effectiveness is obtained on the limit of a sequence of distributions where one randomly chosen bidder which the seller cannot identify has valuation 1 with probability $(n(\alpha-r(n,\alpha)))/(1-r(n,\alpha))$ and valuation just below $r(n,\alpha)$ otherwise, and all other bidders have valuations just below $r(n,\alpha)$. When $\alpha^2n(1-\alpha)>1$, worst-case effectiveness is attained on the same environments on which $\mathcal{E}_n^0(\alpha)$ is attained.

Fix some $r\in[0,\alpha]$.[19] By (2) and Lemma 3, the worst-case effectiveness of any distribution $B\in B_{n,\alpha}$ that induces a distribution of the highest valuation G with $R_{FB}(B)=\int_0^1(1-G(x))\,\mathrm{d}x=\beta$ is bounded from below by

$$\frac{R_E(B,r)}{\beta}\geqslant\begin{cases}r\big(1-G(r^-)\big), & r>\dfrac{n\alpha-\beta}{n-1},\\ \dfrac{n\alpha-\beta}{n-1}, & r\leqslant\dfrac{n\alpha-\beta}{n-1}.\end{cases}$$

By Lemma 1, there exists a sequence of distributions on the limit of which $r(1-G(r^-))$ attains its lower bound of $r(\beta-r)/((1-r)\beta)$. It therefore follows that,

$$\frac{R_E(B,r)}{\beta}\geqslant\begin{cases}\dfrac{r(\beta-r)}{(1-r)\beta}, & r>\dfrac{n\alpha-\beta}{n-1},\\ \dfrac{n\alpha-\beta}{n-1}, & r\leqslant\dfrac{n\alpha-\beta}{n-1}.\end{cases}\tag{20}$$

We are interested in the value of $\beta\in[\alpha,1]$ for which the low bound (20) is the lowest. The fact that $r(\beta-r)/((1-r)\beta)$ is increasing in β and that $r(\beta-r)/((1-r)\beta)\leqslant r\leqslant(n\alpha-\beta)/(n-1)$ implies that the lowest value of (20) is obtained at the lowest value of β which satisfies $r>(n\alpha-\beta)/(n-1)$, or at the limit where $\beta=\min\{n\alpha-nr+r,1\}>\alpha$. It follows that for every $r\in[0,\alpha]$, worst-case effectiveness is bounded from below by

$$\begin{cases}\dfrac{rn(\alpha-r)}{(1-r)(n\alpha-nr+r)}, & r\geqslant\dfrac{n\alpha-1}{n-1},\\ \dfrac{n\alpha-1}{n-1}, & r\leqslant\dfrac{n\alpha-1}{n-1}.\end{cases}\tag{21}$$

depending on whether the lowest possible β is obtained on $n\alpha-nr+r$ or 1.[20] We now solve for the reserve price r that maximizes (21). The fact that

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{rn(\alpha-r)}{(1-r)(n\alpha-nr+r)}\right)=\frac{n((n+\alpha-1)r^2-2\alpha nr+n\alpha^2)}{(1-r)^2(n\alpha-nr+r)^2}$$

[19] It is staightforward to verify that for $r>\alpha$ worst-case effectiveness is equal to zero.

[20] Note that $n\alpha-nr+r\leqslant 1$ if and only if $(n\alpha-1)/(n-1)\leqslant r$.

implies that as a function of the reserve price r, $rn(\alpha - r)/((1-r)(n\alpha - nr + r))$ is increasing on the interval $[0, \alpha(n - \sqrt{n - n\alpha})/(n - 1 + \alpha))$, and decreasing on the interval $(\alpha(n - \sqrt{n - n\alpha})/(n - 1 + \alpha), \alpha)$.[21] Worst-case effectiveness is therefore obtained on $r = \alpha(n - \sqrt{(n - n\alpha)})/(n - 1 + \alpha)$ where it is equal to $rn(\alpha - r)/((1-r)(n\alpha - nr + r))$ provided that $\alpha(n - \sqrt{(n - n\alpha)})/(n - 1 + \alpha) \geqslant (n\alpha - 1)/(n - 1)$, or on any $r \in [0, (n\alpha - 1)/(n - 1)]$ where it equals $(n\alpha - 1)/(n - 1)$, otherwise. Finally, inspection of the inequality

$$\frac{\alpha(n - \sqrt{n - n\alpha})}{n - 1 + \alpha} \geqslant \frac{n\alpha - 1}{n - 1}$$

reveals that for $n \geqslant 1$ and $\alpha \in [0, 1]$, it is satisfied if and only if

$$0 \leqslant \alpha \leqslant \frac{1 + \sqrt{4n - 3}}{2n}. \qquad \square$$

Proof of Theorem 3. By Lemma 3, we may restrict our attention to those distributions $B \in B_{n,\alpha}$ that induce a distribution of the second highest valuation that is given by H^*. We show that among all such distributions, the minimal ratio $E_n^{r(B)}(\alpha; x_2 \sim H^*) = (n\alpha - \beta)/((n - 1)\beta)$ is obtained on a distribution that induces a distribution of the highest valuation that is a truncated Pareto distribution F_ε as in (12) where $\varepsilon = (n\alpha - \beta)/(n - 1)$ and $\beta \in [0, 1]$ satisfies (11), and all other valuations are equal to ε.

By Lemma 2, for every distribution G with $\int_0^1 (1 - G(x))\,\mathrm{d}x = \beta$,

$$\mathcal{E}_1^{r(B)}(\beta) = \frac{\max_{r \in [0,1]}\{r(1 - G(r^-))\}}{\beta} \geqslant \frac{\max_{r \in [0,1]}\{r(1 - F_\varepsilon(r))\}}{\beta} = \frac{\varepsilon}{\beta}$$

where ε is such that $\varepsilon(1 - \log(\varepsilon)) = \beta$. Therefore, for every distribution $B \in \mathcal{B}_{n,\alpha}$ that induces a distribution of the highest valuation G with $\int_0^1 (1 - G(x))\,\mathrm{d}x = \beta$,

$$\begin{aligned} \frac{\max_{r \in [0,1]}\{R_E(B, r)\}}{\beta} &\geqslant \max\left\{\frac{\max_{r \in [0,1]}\{r(1 - G(r^-))\}}{\beta}, \frac{\int_0^1 (1 - H^*(x))\,\mathrm{d}x}{\beta}\right\} \\ &\geqslant \max\left\{\frac{\varepsilon}{\beta}, \frac{n\alpha - \beta}{(n - 1)\beta}\right\}, \end{aligned} \tag{22}$$

where $\varepsilon \in [0, 1]$ is such that

$$\varepsilon\bigl(1 - \log(\varepsilon)\bigr) = \beta. \tag{23}$$

The first term in the right-hand side of (22) corresponds to the revenue obtained from the bidder with the highest valuation when the reserve price is r and the second term corresponds to the revenue obtained from the bidder with the second highest valuation (i.e., when $r = 0$). Because, for $\varepsilon \in (0, 1)$, $\varepsilon(1 - \log(\varepsilon))$ is increasing in ε, ε and therefore also $\varepsilon/\beta = 1/(1 - \log(\varepsilon))$ is increasing in β. On the other hand, $(n\alpha - \beta)/((n - 1)\beta)$ is decreasing in β. Therefore, $\min_{\beta \in [\alpha,1]}\{\max\{\varepsilon/\beta, (n\alpha - \beta)/((n - 1)\beta)\}\}$ is obtained at a β that satisfies[22]

$$\varepsilon = \frac{n\alpha - \beta}{n - 1} \tag{24}$$

and by plugging (17) back into (16) it follows that the minimum ratio is obtained at a $\beta \in [0, 1]$ that satisfies

$$\left(\frac{n\alpha - \beta}{n - 1}\right)\left(1 - \log\left(\frac{n\alpha - \beta}{n - 1}\right)\right) = \beta. \qquad \square$$

Proof of Theorem 4. Fix an $n \geqslant 2$ and an $\alpha \in (0, 1)$. Consider any belief $B \in \mathcal{B}_{n,\alpha}^{ciid}$. By assumption, $B(v_1, \ldots, v_n) = \int \prod_{i=1}^n F(v_i, z)\,\mathrm{d}Z(z)$ for every $(v_1, \ldots, v_n) \in [0, 1]^n$ where for every z, $F(\cdot, z) \in \mathcal{B}_{1,\alpha(z)}$, and

[21] The numbers $\alpha(n - \sqrt{n - n\alpha})/(n - 1 + \alpha)$ and $\alpha(n + \sqrt{n - n\alpha})/(n - 1 + \alpha)$ are the two roots of the equation $(n + \alpha - 1)r^2 - 2\alpha nr + n\alpha^2 = 0$. It is staightforward to verify that $\alpha < \alpha(n + \sqrt{n - n\alpha})/(n - 1 + \alpha)$. Note also that $rn(\alpha - r)/((1 - r)(n\alpha - nr + r)) = (n\alpha - 1)/(n - 1)$ when $r = (n\alpha - 1)/(n - 1)$.

[22] $\beta \geqslant \alpha$ follows from the fact that when $\beta = \alpha$, $(n\alpha - \alpha)/((n - 1)\alpha) = 1 > \varepsilon/\alpha$ because $\varepsilon/\alpha = 1/(1 - \log(\varepsilon)) < 1$ since $\varepsilon \in (0, 1]$.

$\int \alpha(z)\,\mathrm{d}Z(z) \geqslant \alpha$. We show that unless every $F(\cdot, z)$ is a "two-step" function as in (15), the minimal ratios $\mathcal{E}_n^{0,ciid}(\alpha)$ and $\mathcal{E}_n^{r(n,\alpha),ciid}(\alpha)$ cannot be obtained on it. The idea of the proof is to show that unless every $F(\cdot, z)$ is a two-step function, the distribution B can be changed into another distribution $\widehat{B} \in \mathcal{B}_{n,\alpha}^{ciid}$ such that

$$\frac{R_E(B,r)}{R_{FB}(B)} > \frac{R_E(\widehat{B},r)}{R_{FB}(\widehat{B})} \tag{25}$$

for every $r \in [0,1)$. Because every distribution function $B \in \mathcal{B}_{n,\alpha}^{ciid}$ can be arbitrarily closely approximated by a distribution function in $\mathcal{B}_{n,\alpha}^{ciid}$ that describes the distribution of n random variables that are i.i.d. conditional on a random variable that obtains only finitely many values, it is sufficient to prove the theorem only for every such function. That is, we may restrict our attention to distribution functions $B \in \mathcal{B}_{n,\alpha}^{ciid}$ that may be written as $B(v_1,\ldots,v_n) = \sum_{j=1}^m c_j \prod_{i=1}^n F_j(v_i)$ for every $(v_1,\ldots,v_n) \in [0,1]^n$ where $F_j \in \mathcal{B}_{1,\alpha_j}$ for every $j \in \{1,\ldots,m\}$, $\sum_{j=1}^m c_j\alpha_j \geqslant \alpha$, and $(c_1,\ldots,c_m)$ is a vector of positive weights such that $\sum_{j=1}^m c_j = 1$. For every such distribution function, the distribution of the highest and second highest bidders' valuations are given by

$$G(v) = \sum_{j=1}^m c_j\big(F_j(v)\big)^n \quad \text{and} \quad H(v) = \sum_{j=1}^m c_j\Big(n\big(F_j(v)\big)^{n-1} - (n-1)\big(F_j(v)\big)^n\Big),$$

for every $v \in [0,1]$, respectively.

Let $\phi : [0,1] \to [-1,1]$ be a smooth bounded function such that $\widehat{F}_j = F_j + \delta\phi \in \mathcal{B}_{1,\alpha_j}$ for some $\alpha_j \in (0,1]$. Application of (1) and (2), respectively, yields,

$$\begin{aligned} R_{FB}\big((\widehat{F}_j)^n\big) &= \int_0^1 \big[1 - \big(F_j(x) + \delta\phi(x)\big)^n\big]\,\mathrm{d}x = \int_0^1 \left[1 - \sum_{i=0}^n \delta^{n-i}\binom{n}{i}\big(F_j(x)\big)^i\big(\phi(x)\big)^{n-i}\right]\mathrm{d}x \\ &= \int_0^1 \big[1 - \big(F_j(x)\big)^n\big]\,\mathrm{d}x - \delta n\int_0^1 \big(F_j(x)\big)^{n-1}\phi(x)\,\mathrm{d}x \pm O\big(\delta^2\big) \\ &= R_{FB}\big((F_j)^n\big) - \delta n\int_0^1 \big(F_j(x)\big)^{n-1}\phi(x)\,\mathrm{d}x \pm O\big(\delta^2\big) \end{aligned} \tag{26}$$

where $O(\delta^2)$ denotes order of magnitude δ^2,[23] and

$$\begin{aligned} R_E\big((\widehat{F}_j)^n, r\big) &= r\big(1 - \big(F_j(r^-) + \delta\phi(r^-)\big)^n\big) \\ &\quad + \int_r^1 \big[1 - n\big(F_j(x) + \delta\phi(x)\big)^{n-1} + (n-1)\big(F_j(x) + \delta\phi(x)\big)^n\big]\,\mathrm{d}x \\ &= r\big(1 - \big(F_j(r^-)\big)^n\big) - \delta n\phi(r^-)\big(F_j(r^-)\big)^{n-1} \\ &\quad + \int_r^1 \big[1 - n\big(F_j(x)\big)^{n-1} + (n-1)\big(F_j(x)\big)^n\big]\,\mathrm{d}x \\ &\quad - \delta n(n-1)\int_r^1 \phi(x)\big(F_j(x)\big)^{n-2}\big(1 - F_j(x)\big)\,\mathrm{d}x \pm O\big(\delta^2\big) \\ &= R_E\big((F_j)^n, r\big) - \delta n\phi(r^-)\big(F_j(r^-)\big)^{n-1} \\ &\quad - \delta n(n-1)\int_r^1 \phi(x)\big(F_j(x)\big)^{n-2}\big(1 - F_j(x)\big)\,\mathrm{d}x \pm O\big(\delta^2\big). \end{aligned} \tag{27}$$

[23] A function $h(\delta)$ is of an order of magnitude δ^k, denoted $O(\delta^k)$, if $\lim_{\delta\searrow 0} h(\delta)/\delta^k$ is finite.

Lemma 4. *For every constant $C \in [0,1]$, and every cumulative distribution function $F \in B_{1,\alpha}$, if F is not a two-step function as in* (15)*, then there exist two numbers $0 \leqslant a < b < 1$ such that $F(a) < F(b)$, and*

$$\frac{(n-1)((F(b))^{n-2}(1-F(b)) - (F(a))^{n-2}(1-F(a)))}{(F(b))^{n-1} - (F(a))^{n-1}} \neq C. \tag{28}$$

If $r > 0$, then numbers $0 \leqslant a < r < b < 1$ can also be chosen such that $F(a) < F(b)$, and

$$\frac{(n-1)(F(b))^{n-2}(1-F(b))}{(F(b))^{n-1} - (F(a))^{n-1}} \neq C. \tag{29}$$

Proof. By definition, if F is not a two-step function then it assumes at least three different values on the interval $[0,1)$. It follows that there exist three numbers $0 \leqslant x_1 < x_2 < x_3 < 1$ such that $0 \leqslant F(x_1) < F(x_2) < F(x_3) \leqslant 1$. For $n = 2$, the left-hand side of (28) is equal to -1 for any $F(a) < F(b)$.[24] Since for $n \geqslant 3$ the left-hand side of (28) is strictly decreasing in $F(b)$ on the interval $[0,1]$, it must have at least two different valuations, and the conclusion follows.

Suppose now that $0 < r < 1$. If F is not a two-step function then there exist three numbers $0 \leqslant x_1 < x_2 < x_3 < 1$ where $x_1 < r$, $x_2 \neq r$, and $x_3 > r$ such that $0 \leqslant F(x_1) < F(x_2) < F(x_3) \leqslant 1$. Since the left-hand side of (29) is strictly decreasing in $F(b)$ on the interval $[0,1]$, we may choose $0 \leqslant a < r < b < 1$ to satisfy (29). □

Note that for every distribution function $F_j \in \mathcal{B}_{1,\alpha_j}$, and for every two numbers $0 \leqslant a < b < 1$, such that $F_j(a) < F_j(b)$, there exist two non-overlapping intervals of length $l > 0$, I_a and I_b, respectively, such that the average value of F_j on I_a is $F_j(a)$, and the average value of F_j on I_b is $F_j(b)$.

Suppose now that $r = 0$ and that one of the F_j's is not a two-step function. Then, as the previous lemma shows, two numbers a and b can be chosen to satisfy (28) for every constant C, and in particular for $C = R_E(B, r=0)/R_{FB}(B)$. Distinguish between the following two possibilities:

(1) there exist two numbers $0 \leqslant a < b < 1$, such that

$$\frac{(n-1)((F_j(b))^{n-2}(1-F_j(b)) - (F_j(a))^{n-2}(1-F_j(a)))}{(F_j(b))^{n-1} - (F_j(a))^{n-1}} > \frac{R_E(B, r=0)}{R_{FB}(B)}; \tag{30}$$

(2) there exist two numbers $0 \leqslant a < b < 1$, such that

$$\frac{(n-1)((F_j(b))^{n-2}(1-F_j(b)) - (F_j(a))^{n-2}(1-F_j(a)))}{(F_j(b))^{n-1} - (F_j(a))^{n-1}} < \frac{R_E(B, r=0)}{R_{FB}(B)}. \tag{31}$$

In case (1), if $F_j(a) > 0$, then let $\phi : [0,1] \to [-1,1]$ be a smooth function that approximates a function that is equal to -1 on I_a, 1 on I_b, and zero otherwise; and if $F_j(a) = 0$, then let $\phi : [0,1] \to [-1,1]$ be a smooth function that approximates a function that is equal to 1 on I_b, and zero otherwise. In case (2), let $\phi : [0,1] \to [-1,1]$ be a smooth function that approximates a function that is equal to 1 on I_a, -1 on I_b, and zero otherwise. Note that in every case above, ϕ can be chosen so that $\widehat{F}_j = F_j + \delta\phi \in \mathcal{B}_{1,\alpha_j}$ for every small enough δ.

It can be readily verified that for every four real numbers $A, B > 0$, and $x, y \geqslant 0$, such that $x < A$ and $y < B$,

$$\frac{A-x}{B-y} < \frac{A}{B} \quad \Leftrightarrow \quad \frac{x}{y} > \frac{A}{B}, \tag{32}$$

and

$$\frac{A+x}{B+y} < \frac{A}{B} \quad \Leftrightarrow \quad \frac{x}{y} < \frac{A}{B}. \tag{33}$$

Define $\widehat{F}_j = F_j + \delta\phi$ with $\delta > 0$ small enough so that $\widehat{F}_j \in \mathcal{B}_{1,\alpha_j}$. Consider case (1). Note that,

$$\frac{R_E(\widehat{B}, r=0)}{R_{FB}(\widehat{B})} = \frac{\sum_{k \neq j} c_k R_E((F_k)^n, r=0) + c_j R_E((\widehat{F}_j)^n, r=0)}{\sum_{k \neq j} c_k R_{FB}((F_k)^n) + c_j R_{FB}((\widehat{F}_j)^n)}$$

[24] As will become clear at the completion of the proof, this implies that for $n = 2$, $\mathcal{E}_2^{0,ciid}(\alpha)$ is obtained on mixtures of i.i.d distributions with support on 0 and 1.

which by (26) and (27), is equal to,

$$\frac{\sum_{j=1}^{m} c_j R_E((F_j)^n, r=0) - \delta n(n-1)\int_0^1 \phi(x)(F_j(x))^{n-2}(1-F_j(x))\,\mathrm{d}x \pm O(\delta^2)}{\sum_{j=1}^{m} c_j R_{FB}((F_j)^n) - \delta n\int_0^1 (F_j(x))^{n-1}\phi(x)\,\mathrm{d}x \pm O(\delta^2)}.$$

Therefore, by definition of ϕ, $R_E(\widehat{B}, r=0)/R_{FB}(\widehat{B})$ is approximately equal to,

$$\frac{\sum_{j=1}^{m} c_j R_E((F_j)^n, r=0) - \delta n(n-1)((F_j(b))^{n-2}(1-F_j(b)) - (F_j(a))^{n-2}(1-F_j(a))) \pm O(\delta^2)}{\sum_{j=1}^{m} c_j R_{FB}((F_j)^n) - \delta n((F_j(b))^{n-1} - (F_j(a))^{n-1}) \pm O(\delta^2)}.$$

Finally, the fact that for small δ, terms of order of magnitude δ^2 may be ignored, together with (30) and (32), imply (25). Consider now case (2). As before, by (26) and (27),

$$\frac{R_E(\widehat{B}, r=0)}{R_{FB}(\widehat{B})} = \frac{\sum_{j=1}^{m} c_j R_E((F_j)^n, r=0) - \delta n(n-1)\int_0^1 \phi(x)(F_j(x))^{n-2}(1-F_j(x))\,\mathrm{d}x \pm O(\delta^2)}{\sum_{j=1}^{m} c_j R_{FB}((F_j)^n) - \delta n\int_0^1 (F_j(x))^{n-1}\phi(x)\,\mathrm{d}x \pm O(\delta^2)}.$$

Therefore, by definition of ϕ, $R_E(\widehat{B}, r=0)/R_{FB}(\widehat{B})$ is approximately equal to,

$$\frac{\sum_{j=1}^{m} c_j R_E((F_j)^n, r=0) + \delta n(n-1)((F_j(b))^{n-2}(1-F_j(b)) - (F_j(a))^{n-2}(1-F_j(a))) \pm O(\delta^2)}{\sum_{j=1}^{m} c_j R_{FB}((F_j)^n) + \delta n((F_j(b))^{n-1} - (F_j(a))^{n-1}) \pm O(\delta^2)}.$$

As before, for small δ, terms of order of magnitude δ^2 may be ignored, together with (31) and (33), this implies (25).

The proof for the case where $r > 0$ employs a similar idea. Suppose that $r > 0$ and that one of the F_j's is not a two-step function. By Lemma 4 there exist two numbers $a < r < b$ that satisfy (29) for every constant C, and in particular for $C = R_E(B, r)/R_{FB}(B)$. Distinguish between the following two possibilities:

(1) the two numbers $0 \leqslant a < r < b < 1$ are such that

$$\frac{(n-1)(F_j(b))^{n-2}(1-F_j(b))}{(F_j(b))^{n-1} - (F_j(a))^{n-1}} > \frac{R_E(B, r)}{R_{FB}(B)}; \tag{34}$$

(2) there exist two numbers $0 \leqslant a < r < b < 1$, such that

$$\frac{(n-1)(F_j(b))^{n-2}(1-F_j(b))}{(F_j(b))^{n-1} - (F_j(a))^{n-1}} < \frac{R_E(B, r=0)}{R_{FB}(B)}. \tag{35}$$

In case (1), if $F_j(a) > 0$, then let $\phi : [0,1] \to [-1,1]$ be a smooth function that approximates a function that is equal to -1 on I_a, 1 on I_b, and zero otherwise; and if $F_j(a) = 0$, then let $\phi : [0,1] \to [-1,1]$ be a smooth function that approximates a function that is equal to 1 on I_b, and zero otherwise. In case (2), let $\phi : [0,1] \to [-1,1]$ be a smooth function that approximates a function that is equal to 1 on I_a, -1 on I_b, and zero otherwise. Note that in every case above, ϕ can be chosen so that $\widehat{F}_j = F_j + \delta\phi \in \mathcal{B}_{1,\alpha_j}$ for every small enough δ.

Define $\widehat{F}_j = F_j + \delta\phi$ with $\delta > 0$ small enough so that $\widehat{F}_j \in \mathcal{B}_{1,\alpha_j}$. Consider case (1). Note that by definition of ϕ, $R_E(\widehat{B}, r)/R_{FB}(\widehat{B})$ is approximately equal to,

$$\frac{R_E(\widehat{B}, r)}{R_{FB}(\widehat{B})} = \frac{\sum_{j=1}^{m} c_j R_E((F_j)^n, r) - \delta n(n-1)(F_j(b))^{n-2}[1-F_j(b)] \pm O(\delta^2)}{\sum_{j=1}^{m} c_j R_{FB}((F_j)^n) - \delta n((F_j(b))^{n-1} - (F_j(a))^{n-1}) \pm O(\delta^2)}.$$

As before, the fact that for small δ, terms of order of magnitude δ^2 may be ignored, together with (34) and (32), imply (25). Consider now case (2). As before, the definition of ϕ implies that $R_E(\widehat{B}, r)/R_{FB}(\widehat{B})$ is approximately equal to,

$$\frac{R_E(\widehat{B}, r=0)}{R_{FB}(\widehat{B})} = \frac{\sum_{j=1}^{m} c_j R_E((F_j)^n, r=0) + \delta n(n-1)(F_j(b))^{n-2}(1-F_j(b)) \pm O(\delta^2)}{\sum_{j=1}^{m} c_j R_{FB}((F_j)^n) + \delta n((F_j(b))^{n-1} - (F_j(a))^{n-1}) \pm O(\delta^2)}$$

which, for small δ, together with (35) and (33), implies (25). Finally, we write that worst-case effectiveness is obtained on a limit of a sequence of mixtures of i.i.d. two-step distributions because of reasons similar to those in the proof of Lemma 1, namely, for similar considerations, the "jump" in F may occur "just before" the reserve price $r(n,\alpha)$. □

Proof of Theorem 5. Fix an $n \geqslant 2$ and an $\alpha \in (0,1]$. Consider any environment $B \in \mathcal{B}_{n,\alpha}$. Let G denote the corresponding distribution of $\max\{v_1,\dots,v_n\}$. The expected revenue under the posted-price mechanism is given by

$$\max_{r\in[0,1]}\bigl\{r\bigl(1-G(r^-)\bigr)\bigr\}$$

where $G(r^-)=\lim_{x\nearrow r} G(x)$. The effectiveness of the posted-price mechanism in the environment B is therefore given by

$$\max_{r\in[0,1]}\left\{\frac{r(1-G(r^-))}{\int_0^1(1-G(x))\,\mathrm{d}x}\right\}. \tag{36}$$

Suppose that $\int_0^1(1-G(x))\,\mathrm{d}x=\beta\geqslant\alpha$. By Lemma 2, for every such β,

$$\max_{r\in[0,1]}\left\{\frac{r(1-G(r^-))}{\int_0^1(1-G(x))\,\mathrm{d}x}\right\}\geqslant\frac{1}{1-\log(\varepsilon)}$$

where $\varepsilon\in[0,1]$ is the unique solution to $\varepsilon(1-\log(\varepsilon))=\beta$. Because, for $\varepsilon\in(0,1)$, $\varepsilon(1-\log(\varepsilon))$ is increasing in ε, ε and therefore also $1/(1-\log(\varepsilon))$ is increasing in β. Therefore, the minimum of $1/(1-\log(\varepsilon))$ is obtained at $\beta=\alpha$. In fact, the inequality in (36) is binding at the environment where all the buyers' valuations are identical and distributed according to the truncated Pareto distribution F_ε where $\varepsilon(1-\log(\varepsilon))=\alpha$. It therefore follows that $\mathcal{E}^{PP}(\alpha)=1/(1-\log(\varepsilon))$ where $\varepsilon\in[0,1]$ is the unique solution to $\varepsilon(1-\log(\varepsilon))=\alpha$. □

Calculating $\mathcal{E}_n^{0,ciid}(\alpha)$ *and* $\mathcal{E}_n^{r(n,\alpha),ciid}(\alpha)$. Fix some $n\geqslant 3$ and $\alpha\in(0,1]$. By Theorem 4, we know that $\mathcal{E}_n^{0,ciid}(\alpha)$ and $\mathcal{E}_n^{r(n,\alpha),ciid}(\alpha)$ are obtained on distributions that are mixtures of i.i.d. two-step distribution functions. For every $r\in[0,1)$, we solve numerically for the particular two-step functions that attain worst-case effectiveness on i.i.d. environments with n bidders and expected valuations α_j, $j\in J$. Denote these functions by $\{F_{\alpha_j}\}_{j\in J}$. As noted in the text, all these two-step functions have a first step that is equal to zero. We then numerically solve for,

$$\max_{r\in[0,1]}\left\{\min_{\{\lambda_j\}_{j\in J}}\left\{\frac{\sum_{j\in J}\lambda_j R_E((F_{\alpha_j})^n,r)}{\sum_{j\in J}\lambda_j R_{FB}([F_{\alpha_j}]^n)}\right\}\right\}$$

subject to

$$\sum_{j\in J}\lambda_j\alpha_j\geqslant\alpha,\qquad \sum_{j\in J}\lambda_j=1,$$

and

$$\lambda_j\geqslant 0\quad\text{for every } j\in J.$$

As noted in the text, remarkably, the minimum is obtained on degenerate mixtures, namely, for every $\alpha_j\in(0,1]$, the minimum is obtained on the vector $\{\lambda_k\}_{k\in J}$ where $\lambda_j=1$ and $\lambda_k=0$ for every $k\neq j$.

We also conducted robustness checks to verify that replacing the two-step distribution functions that attain worst-case effectiveness for i.i.d. environments by other two-step distribution functions and minimizing subject to the constraints above only increases the value of the objective function.

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